Math 511: Linear Algebra
Final Exam, Part I

Thursday, 24 July 2014

Name: Key

Instructions: Complete all 3 problems in part I, and 3 of the 4 problems in part II. Clearly mark the problem in part II that you would like to omit. Each problem in part I is worth 20 points; each completed problem in part II is worth 15 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

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Part I. Complete all 3 problems in the space provided. Show enough work. Each problem is worth 20 points.

1. Consider the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$ of $\mathbb{R}^2$ where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Use the Gram-Schmidt process to find an orthonormal basis of $\mathbb{R}^2$.

$$\underline{u}_1 = \frac{\underline{x}_1}{\|\underline{x}_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \underline{u}_1}$$

$$p_1 = \langle \underline{x}_2, \underline{u}_1 \rangle \underline{u}_1 = \frac{1}{\sqrt{2}} \cdot 2 \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$x_2 - p_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{u}_2 = \frac{x_2 - p_1}{\|x_2 - p_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \underline{u}_2}$$

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2. Write the matrix A as a product $X D X^{-1}$ where D is diagonal and X is nonsingular.

$$A = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$$

Clearly identify X, D, and X^{-1}.

evalues: $\begin{vmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{vmatrix} = (\lambda+1)(\lambda-4)+6$

$= \lambda^2 - 3\lambda - 4 + 6$

$= \lambda^2 - 3\lambda + 2$

$= (\lambda-2)(\lambda-1)$

so $\lambda_1 = 1$, $\lambda_2 = 2$

$$\boxed{D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}$$

$\underline{x}_1$: $N(A-I)$: $\left(\begin{array}{cc|c} -2 & 6 & 0 \\ -1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad \underline{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\underline{x}_2$: $N(A-2I)$: $\left(\begin{array}{cc|c} -3 & 6 & 0 \\ -1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad \underline{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\boxed{X = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}}$$

$$\boxed{X^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}}$$

and

$$\boxed{A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}}$$

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3. Prove the theorems.

(*a.*) **Theorem.** *Let* $\langle\ ,\ \rangle$ *be the standard Euclidean inner product on* $\mathbb{R}^n$*; then*

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

In $\mathbb{R}^n$, $\langle x, y\rangle = \|x\|\,\|y\| \cos\theta$ by Law of Cosines argument.

Then $|\langle x,y\rangle| = \big|\, \|x\|\,\|y\| \cos\theta \,\big|$

$= \|x\|\,\|y\|\, |\cos\theta|$

where $|\cos\theta| \leq 1$

$\Rightarrow |\langle x,y\rangle| \leq \|x\|\,\|y\|.$ $\square$

(*b.*) **Theorem.** *Let* $(V, \langle\ ,\ \rangle)$ *be an inner product space. If* $\mathbf{x} \perp \mathbf{y}$*, then*

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$

where $\|x\|^2 = \langle x,x\rangle$, $\|y\|^2 = \langle y,y\rangle$, and $\langle x,y\rangle = 0$ since $x \perp y$.

Thus

$\|x+y\|^2 = \|x^2\| + \|y\|^2.$ $\square$

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Part II. Complete 3 of the 4 problems. Show enough work. Clearly mark the one problem that you wish to omit. Each completed problem is worth 15 points.

4. Consider the initial value problem

$$\begin{cases} y'' - 7y' + 12y = 0; \\ y(0) = 1, \\ y'(0) = -1. \end{cases}$$

Solve the IVP by reducing it to a system of first-order differential equations. Clearly label the solution $y = y(t)$.

Put $y_1 = y$, $y_2 = y'$; $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -12y_1 + 7y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix} Y$

so $Y' = AY$

e'values of A: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -12 & 7-\lambda \end{vmatrix} = +\lambda(\lambda - 7) + 12 = \lambda^2 - 7\lambda + 12$
$= (\lambda - 3)(\lambda - 4)$

so $\lambda_1 = 3$, $\lambda_2 = 4$

$X_1 \in N(A - 3I)$: $\left(\begin{array}{cc|c} -3 & 1 & 0 \\ -12 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$X_2 \in N(A - 4I)$: $\left(\begin{array}{cc|c} -4 & 1 & 0 \\ -12 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 4 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

so $X = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$ and $X^{-1} = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}$

$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = X^{-1} Y(0) = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

so $Y(t) = 5e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 4e^{4t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5e^{3t} - 4e^{4t} \\ 15e^{3t} - 16e^{4t} \end{pmatrix}$

and $\boxed{y(t) = 5e^{3t} - 4e^{4t}}$

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5. Consider the vector space P_3.

10 (*a.*) Show that

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$$

defines an inner product on P_3.

1.) $\langle p,p\rangle = \int_0^1 (p(x))^2\,dx \geq 0$

and $\langle p,p\rangle = \int_0^1 (p(x))^2\,dx = 0$ iff $p(x)=0$.

2.) $\langle p,q\rangle = \int_0^1 p(x)q(x)\,dx = \int_0^1 q(x)p(x)\,dx = \langle q,p\rangle$.

3. $\langle \alpha p+\beta q, r\rangle = \int_0^1 (\alpha p(x) + \beta q(x))\, r(x)\,dx = \int_0^1 \alpha p(x) r(x)\,dx + \int_0^1 \beta q(x) r(x)\,dx$

$= \alpha \int_0^1 p(x) r(x)\,dx + \beta \int_0^1 q(x) r(x)\,dx = \alpha\langle p,r\rangle + \beta\langle q,r\rangle$. $\square$

5 (*b.*) Find the projection $\mathbf{proj}_q p$ of $p(x) = x^2$ onto $q(x) = x$ with respect to this inner product.

$$\text{proj}_q p = \frac{\langle p,q\rangle}{\langle p,p\rangle} p = \frac{\langle x^2, x\rangle}{\langle x, x\rangle} x = \frac{\int_0^1 x^3\,dx}{\int_0^1 x^2\,dx} x = \frac{1/3}{1/2} x = \boxed{\frac{2}{3}x}.$$

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6. Recall that the 1-norm on $\mathbb{R}^3$ is defined by

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + |x_3|,$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$.

5 (*a.*) Show that this formula indeed defines a norm.

1. $\|x\|_1 = |x_1| + |x_2| + |x_3| \geq 0$ and $=0$ iff $x_1 = x_2 = x_3 = 0 \Rightarrow \underline{x} = \underline{0}$.

2. $\|x+y\|_1 = |x_1+y_1| + |x_2+y_2| + |x_3+y_3|$

$\leq |x_1| + |y_1| + |x_2| + |y_2| + |x_3| + |y_3|$

$= (|x_1| + |x_2| + |x_3|) + (|y_1| + |y_2| + |y_3|)$

$= \|x\|_1 + \|y\|_1$. $\square$

10 (*b.*) Show that this norm is <u>not</u> derived from an inner product on $\mathbb{R}^3$.

~~$\|x+y\|_1 = |x_1+y_1| + |x_2+y_2| + |x_3+y_3|$~~

Let $\underline{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\underline{y} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$\|x+y\|_1 = 2+1 = 3$

$\|x-y\|_1 = 0+1 = 1$

$\|x+y\|_1^2 = 9$

$+\|x-y\|_1^2 = 1$

10

$\|x\|_1 = 1$

$\|y\|_1 = 2$

$2\|x\|_1^2 = 2$

$2\|y\|_1^2 = 2 \cdot 4 = 8$

10

Crap!!

$\underline{x} = \underline{e}_1$

$\underline{y} = \underline{e}_2$

$x+y = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $x-y = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$\|x+y\|_1^2 = (2)^2 = 4$

$\|x-y\|_1^2 = (2)^2 = 4$

$4+4=8$

$2\|x\|_1^2 = 2\cdot(1)^2 = 2$

$2\|y\|_1^2 = 2\cdot(1)^2 = 2$

$2+2=4$

$8 \neq 4$!

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7. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}.$$

(*a.*) Show that $A^2 = 0$, where 0 is the zero-matrix.

$$A^2 = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 4-4 & -2+2 \\ 8-8 & -4+4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(*b.*) Compute the exponential of the matrix A, e^A.

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$$

For this A; $A^2, A^3, \ldots = 0$.

So $e^A = I + A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = \boxed{\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} = e^A}$