Math 511: Linear Algebra

Midterm Exam ~~1~~ 2

Thursday, 3 July 2014

Name: Key

Instructions: Complete all 3 problems in Part I, and 3 of the 5 problems in Part II. Clearly mark the two problems that you would like to omit in Part II. If you don't tell me which problems to omit, then I will automatically omit the last 2! No exceptions.

Each problem in Part I is worth 20 points. Each completed problem in Part II is worth 15 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

Name:____________________________________

Part I. Complete all 3 problems in the space provided. Show enough work.

1. Consider the vector space P_3 with standard basis $E = \{1, x, x^2\}$ and ordered basis $V = \{1, (x-1), (x-1)^2\}$.

$\boxed{10}$ (a.) Find the transition matrix from E to V.

$$V = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

so $\boxed{V^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}$

$\boxed{10}$ (b.) Write the vector $p(x) = x^2$ in V-coordinates.

$$p(x) = x^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_E$$

$$[p]_V = V^{-1}p = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_E = \boxed{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_V = [p]_V}$$

$$1 + 2(x-1) + (x-1)^2 = 1 + 2x - 2 + x^2 - 2x + 1 = x^2 \quad ✓$$

Name:________________________________

2. Consider the transformation $L : P_3 \to P_3$ defined by $L(p)(x) = x \cdot p'(x) + p(0)$.

$\boxed{10}$ (*a.*) Prove that L is linear.

$$(L.1.)\quad L(\alpha p) = x \cdot (\alpha p)'(x) + (\alpha p)(0) = \alpha \cdot x p'(x) + \alpha \cdot p(0)$$
$$= \alpha\left(x p'(x) + p(0)\right) = \alpha L(p) \checkmark$$

$$(L.2.)\quad L(p+q) = x \cdot (p+q)'(x) + (p+q)(0) = x \cdot p'(x) + x q'(x) + p(0) + q(0)$$
$$= \left(x p'(x) + p(0)\right) + \left(x q'(x) + q(0)\right) = L(p) + L(q). \quad \square$$

$\boxed{10}$ (*b.*) Find the matrix representing L with respect to the ordered basis $\{1, (x-1), (x-1)^2\}$.

$$L(1) = x \cdot 0 + 1 = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_E$$

$$L(x-1) = x \cdot 1 + -1 = x - 1 = \begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix}_E$$

$$L(x-1)^2 = L(x^2 - 2x + 1) = x \cdot (2x - 2) + 1 = 2x^2 - 2x + 1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}_E$$

$$\text{So}\quad A = V^{-1}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = A}$$

Name:________________________________

20

3. Let L be a linear operator on $\mathbb{R}^2$ such that

$$L\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix},\quad\text{and}\quad L\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-3\\-1\end{pmatrix}.$$

Compute $L\begin{pmatrix}4\\4\end{pmatrix}$.

$$U=\begin{pmatrix}-2&1\\1&-1\end{pmatrix}$$

$$U^{-1}=\begin{pmatrix}-1&-1\\-1&-2\end{pmatrix}$$

let $\underline{x}=\begin{pmatrix}4\\4\end{pmatrix}$

$$[x]_U=U^{-1}\underline{x}=\begin{pmatrix}-1&-1\\-1&-2\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix}=\begin{pmatrix}-8\\-12\end{pmatrix}$$

$$L\begin{pmatrix}4\\4\end{pmatrix}=L\left(-8\cdot\begin{pmatrix}-2\\1\end{pmatrix}-12\cdot\begin{pmatrix}1\\-1\end{pmatrix}\right)=-8L\begin{pmatrix}-2\\1\end{pmatrix}-12\cdot L\begin{pmatrix}1\\-1\end{pmatrix}=-8\begin{pmatrix}2\\4\end{pmatrix}-12\begin{pmatrix}-3\\-1\end{pmatrix}$$

$$=\begin{pmatrix}-16\\-32\end{pmatrix}-\begin{pmatrix}-36\\-12\end{pmatrix}=\begin{pmatrix}-16+36\\-32+12\end{pmatrix}=\begin{pmatrix}20\\-20\end{pmatrix}$$

so $\boxed{L\begin{pmatrix}4\\4\end{pmatrix}=\begin{pmatrix}20\\-20\end{pmatrix}}$

Name: ______________________________

Part II. Complete 3 of the 5 problems. Show enough work. **Clearly mark the two problems that you wish to omit.**

$\boxed{15}$

~5 each

4. Consider the matrix

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 6 & -3 & -5 & -7 \end{pmatrix}.$$

Find bases for the row, column, and null spaces of A. Clearly label each answer.

$$\begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 6 & -3 & -5 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 7 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 5/7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & -9/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 5/7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -5/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 5/7 \end{pmatrix}.$$

$$\boxed{\text{Col}(A) = \text{span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} \right\}.}$$

$$\boxed{\text{Row}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -5/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5/7 \end{pmatrix} \right\}}$$

$N(A)$:
$$\begin{aligned} x_4 &= \alpha \\ x_3 &= -\tfrac{5}{7}\alpha \\ x_2 &= \tfrac{2}{7}\alpha \\ x_1 &= 5/7\,\alpha \end{aligned} \Rightarrow \boxed{N(A) = \text{span}\left\{ \begin{pmatrix} 5 \\ 2 \\ -5 \\ 7 \end{pmatrix} \right\}.}$$

!

Name:______________________________

5. Let $A, B, C \in \mathbb{R}^{n\times n}$, and $\lambda \in \mathbb{R}$. Prove the following results.

$\boxed{7.5}$ (*a*.) If A is similar to B and B is similar to C, then A is similar to C. ok

$A \sim B \Rightarrow \quad SA = BS \quad \Rightarrow \quad A = S^{-1}BS$

$B \sim C \Rightarrow \quad TB = CT \quad \Rightarrow \quad B = T^{-1}CT$

so $\quad A = S^{-1}BS = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS)$

$= (TS)^{-1}C(TS)$

Thus $A \sim C$. $\square$

$\boxed{7.5}$ (*b*.) If A is similar to B, then $A - \lambda I$ is similar to $B - \lambda I$.

$A \sim B \Rightarrow A = S^{-1}BS.$

$S^{-1}(B - \lambda I)S = S^{-1}BS - S^{-1}(\lambda I)S$

$= S^{-1}BS - \lambda S^{-1}IS$

$= A - \lambda I. \quad \square$

Thus $(A - \lambda I) \sim (B - \lambda I)$.

Name:______________________________

6. Consider the following vectors in $\mathbb{R}^2$,

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

ok

$\boxed{7.5}$ (*a*.) Find the transition matrix from the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ to the ordered basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

$V = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ $\qquad S = U^{-1}V$

$U = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

$U^{-1} = \frac{1}{1}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

$S = U^{-1}V = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 10 & 11 \\ 4 & 4 \end{pmatrix} = S}$

$\boxed{7.5}$ (*b*.) Write the vector $\mathbf{x} = 4\mathbf{v}_1 - 2\mathbf{v}_2$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

$[\underline{x}]_V = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$[\underline{x}]_U = S[\underline{x}]_V = \begin{pmatrix} 10 & 11 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix}_V = \begin{pmatrix} 18 \\ 8 \end{pmatrix}_U$

So $\boxed{\underline{x} = 18\underline{u}_1 + 8\underline{u}_2}$

Name:______________________________________

7. Let $A \in \mathbb{R}^{3\times 5}$ with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_5$. Further, suppose $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_5$ are linearly independent, $\mathbf{a}_3 = \mathbf{a}_1 - \mathbf{a}_2$, and $\mathbf{a}_4 = 2\mathbf{a}_1 + \mathbf{a}_5$.

7.5 (*a*.) What is the reduced row echelon form (RREF) of A?

$$\boxed{\mathrm{RREF}(A) = \begin{pmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}$$

7.5 (*b*.) What is the column space of A?

$\mathrm{Col}(A) = \mathrm{span}\{a_1, a_2, a_5\} = \mathbb{R}^3$!

Name:________________________________

$\boxed{15}$ **8.** Let

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and let L be the linear transformation from $\mathbb{R}^4$ into $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = x_4\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 - x_3)\mathbf{b}_3.$$

Find the matrix A representing L with respect to the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$$L(\underline{e}_1) = 0\underline{b}_1 + 0\underline{b}_2 + 1\underline{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_B$$

$$L(\underline{e}_2) = 0\underline{b}_1 + 1\underline{b}_2 + 0\underline{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_B$$

$$L(\underline{e}_3) = 0\underline{b}_1 + 0\underline{b}_2 - 1\underline{b}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}_B$$

$$L(\underline{e}_4) = 1\underline{b}_1 + 0\underline{b}_2 + 0\underline{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B$$

So,

$$\boxed{A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}}$$