Math 511: Linear Algebra
Midterm Exam 1

Friday, 20 June 2014

Name: Key

Instructions: Complete all 4 problems in part I, and 2 of the 3 problems in each of parts II and III. Clearly mark the problem that you would like to omit in each ofparts II and III. Each problem in Part I is worth 15 points. Each completed problem in parts II and III is worth 10 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

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Part I. Complete all 4 problems in the space provided. Show enough work.

1–2. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_4 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}.$$

1. Show that $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ form a basis for $\mathbb{R}^3$.

Put $X = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

$\det(X) = -1 \cdot 1 \cdot -1 = 1$

Since $\det X \neq 0$, its columns form a basis for $\mathbb{R}^3$.

2. Write $\mathbf{x}_4$ as a linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

Solve: $\left(\begin{array}{ccc|c} -1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -3 \end{array}\right)$

$c_1 = -2$

$c_2 = 1 - c_1 = 1+2 = 3$

$+c_3 = 3 + c_2 = 3+3 = 6$

So $\boxed{\underline{x}_4 = -2\underline{x}_1 + 3\underline{x}_2 + 6\underline{x}_3}$

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3. Consider the subset of P_3 given by $S = \{p \in P_3 \mid p(0) = 0 \text{ and } p(2) = 0\}$. Does S form a subspace of P_3? Justify your answer (prove or provide a counterexample).

Let $p, q \in S$ and $\alpha \in \mathbb{R}$.

S1. $\left.\begin{array}{l}(\alpha p)(0) = \alpha(p(0)) = \alpha \cdot 0 = 0 \\ (\alpha p)(2) = \alpha(p(2)) = \alpha \cdot 0 = 0\end{array}\right\}$ so $(\alpha p) \in S$

S2. $\left.\begin{array}{l}(p+q)(0) = p(0) + q(0) = 0+0 = 0 \\ (p+q)(2) = p(2) + q(2) = 0+0 = 0\end{array}\right\}$ so $(p+q) \in S$.

Therefore S $\underline{\underline{\text{is}}}$ a subspace of P_3.

4. Consider the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Compute A^2 and A^3. Find a general formula for A^n.

$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$A^2 = A \cdot A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$A^3 = A^2 \cdot A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$

$\vdots$

$A^n = \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$.

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Part II. Complete 2 of the 3 problems. Show enough work. Clearly mark the one problem that you wish to omit.

5. A matrix $N \in \mathbb{R}^{n\times n}$ is said to be *nilpotent* if $N^k = \mathbf{0}$ for some $k \geq 1$, where $\mathbf{0} \in \mathbb{R}^{n\times n}$ is the zero-matrix.

Prove that if N is nilpotent, then $\det(N) = 0$.

Since $N^k = \underline{0}$

$0 = \det(\underline{0}) = \det(N^k) = (\det(N))^k$

and $\sqrt[k]{(\det(N))^k} = \det(N)$, $\sqrt[k]{0} = 0$.

Therefore $\det(N) = 0$.

6. Let $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that if $A\mathbf{x} = A\mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$, then A is singular.

Suppose A is nonsingular. Then A^{-1} exists and

$A^{-1}A\underline{x} = A^{-1}A\underline{y}$

$\Rightarrow I\underline{x} = I\underline{y}$

$\Rightarrow \underline{x} = \underline{y}$.

But this contradicts the assumption that $\underline{x} \neq \underline{y}$.

Therefore A must be singular. $\square$

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7. Suppose that the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ span a vector space V. Let $\mathbf{v}$ be any other vector in V. Prove that the set $\{\mathbf{v}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ must be linearly dependent.

Since $\{\underline{x}_1, \ldots, \underline{x}_n\}$ span V, then $\underline{v} = c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots + c_n\underline{x}_n$ for some coefficients $c_1, \ldots, c_n \in \mathbb{R}$. Subtracting $\underline{v}$ from both sides,

$$\underline{0} = c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots + c_n\underline{x}_n + (-1)\underline{v}.$$

Since the coefficient on $\underline{v}$ is not 0, the set $\{\underline{v}, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\}$ is linearly dependent. D

Part III. Complete 2 of the 3 problems. Show enough work. Clearly mark the one problem that you wish to omit.

8. Let S be the subspace of P_3 satisfying $p(0) = 0$ and $p(2) = 0$. Find a basis for S.

let $p(x) = a + b(x-2) + c(x-2)^2$ be the representation for $p \in P_3$ centered at 2.

Then for $p \in S$, we must have $a = 0$.

Now $p(0) = -2b + 4c$

For $p \in S$, then $-2b + 4c = 0$, or $b = 2c$.

$$\text{Thus } p(x) = 2c(x-2) + c(x-2)^2 = 2cx - 4c + cx^2 - 4cx + 4c$$
$$= cx^2 - 2cx$$
$$= c(x^2 - 2x)$$

so $S = \operatorname{span}\{(x^2-2x)\}$, or $p(x) = x^2 - 2x$ is a basis for S.

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9. Find the null space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 & 2 \\ 3 & 3 & -1 & -3 & -5 \\ -2 & -2 & 2 & 1 & -4 \end{pmatrix}.$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 2 & 0 \\ 3 & 3 & -1 & -3 & -5 & 0 \\ -2 & -2 & 2 & 1 & -4 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \end{array}\right)$$

$3R_1 - R_2 \to R_2$

$2R_1 + R_3 \to R_3$

$2R_2 - R_3 \to R_3$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & \boxed{1} & 0 & 0 & \boxed{4} & 0 \\ 0 & \boxed{0} & 1 & 0 & \boxed{1} & 0 \\ 0 & \boxed{0} & 0 & 1 & \boxed{2} & 0 \end{array}\right)$$

$R_1 + R_3 \to R_1$

x_2 x_5 are free.

$x_1 = -\beta - 4\alpha$

$x_2 = \beta$

$x_3 = -\alpha$

$x_4 = -2\alpha$

$x_5 = \alpha$

so $\underline{x} \in N(A)$ looks like

$$\underline{x} = \beta\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} -4 \\ 0 \\ -1 \\ -2 \\ 1 \end{pmatrix}$$

or $\boxed{N(A) = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -1 \\ -2 \\ 1 \end{pmatrix}\right\}}$

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10. Consider a linear system whose matrix equation is of the form

$$\begin{pmatrix} 1 & -4 & -3 \\ 2 & 4 & 2 \\ -2 & 2 & \alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}.$$

For what values of α will the system have a unique solution?

System has a unique sol'n iff $\det(A) \neq 0$.

Expand on row 3:

$$\det(A) = -2\begin{vmatrix} -4 & -3 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} + \alpha\begin{vmatrix} 1 & -4 \\ 2 & 4 \end{vmatrix} \neq 0$$

or

$$\alpha \neq \frac{2(-8+12)+2(2+6)}{4+8} = \frac{8+16}{12} = \frac{24}{12} = 2.$$

Thus the system has a unique solution for all values of $\boxed{\alpha \neq 2.}$