6.2 and 6.3 Differential Equations and Diagonalizable matrices

Eigenvalues can be used to help solve linear systems of DE.

Consider a system of the form

$$\begin{aligned} y_1' &= a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n \\ y_2' &= a_{21} y_1 + a_{22} y_2 + \cdots + a_{2n} y_n \\ &\vdots \\ y_n' &= a_{n1} y_1 + a_{n2} y_2 + \cdots + a_{nn} y_n \end{aligned}$$

where $y_i \in C^1[a,b]$ for each i. If we let

$$Y = (y_1, y_2, \ldots, y_n)^T \quad \text{and} \quad Y' = (y_1', y_2', \ldots, y_n')^T$$

then the system can be written as

$$Y' = AY$$

where Y' and Y can both be thought of as functions of a single real variable t.

When $n = 1$, we obtain $y' = ay$.

We know that the solution is $y = ce^{at}$ $c \in \mathbb{R}$.

A natural generalization of this then is to guess that

$$Y = e^{\lambda t} \underline{x} \quad \text{for} \quad \underline{x} \in \mathbb{R}^n.$$

Then $Y' = \lambda e^{\lambda t} \underline{x} = \lambda (e^{\lambda t} \underline{x}) = \lambda Y.$

Now if we choose λ to be an eigenvalue of A, and $\underline{x}$ to be a corresponding eigen vector, then

$$AY = e^{\lambda t} A \underline{x} = \lambda e^{\lambda t} \underline{x} = \lambda Y = Y'$$

Hence, $Y = e^{\lambda t} \underline{x}$ is a solution of the system!

This holds for any eigenvalue, eigenvector combination. Further notice that if Y_1 and Y_2 are two solutions, then $\alpha Y_1 + \beta Y_2$ is also a solution.

RE. Verify this.

It follows that if $Y_1, \dots, Y_n$ are any solutions, then
$$c_1 Y_1 + c_2 Y_2 + \dots + c_n Y_n$$
is again a solution.

In general, the solutions of an $n \times n$ system of first-order equations forms an n-dim vector space of continuous vector-valued functions.

However, if we prescribe an initial condition $Y(0) = Y_0$, then the IVP will have a unique solution.

Ex. Solve the system

$$y_1' = 3y_1 + 4y_2$$
$$y_2' = 3y_1 + 2y_2$$

$$Y' = AY \quad \text{where} \quad A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$$

e'values of A are $\lambda_1 = 6$ and $\lambda_2 = -1$. E'vectors are $\underline{x}_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\underline{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Thus $Y = c_1 e^{\lambda_1 t} \underline{x}_1 + c_2 e^{\lambda_2 t} \underline{x}_2$

$$= c_1 e^{6t}\begin{pmatrix}4\\3\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}4e^{6t} & e^{-t}\\ 3e^{6t} & -e^{-t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = Y$$

for any $\underline{c} = \begin{pmatrix}c_1\\c_2\end{pmatrix} \in \mathbb{R}^2$.

Now suppose $Y(0) = \begin{pmatrix}6\\1\end{pmatrix}$

Then $Y(0) = \begin{pmatrix}4 & 1\\3 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}6\\1\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{-1}{7}\begin{pmatrix}-1 & -1\\-3 & 4\end{pmatrix}\begin{pmatrix}6\\1\end{pmatrix}$$

$$= \frac{-1}{7}\begin{pmatrix}-7\\-14\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

So $Y = \begin{pmatrix}4e^{6t} & e^{-t}\\3e^{6t} & -e^{-t}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}4e^{6t} + 2e^{-t}\\3e^{6t} - 2e^{-t}\end{pmatrix} = Y(t)$

solves the IVP! Cool!

Note: If $\lambda \in \mathbb{C}$, then $e^{\lambda t} = e^{at+ibt} = e^{at}e^{ibt}$
$= e^{at}(\cos(bt) + i\sin(bt))$

You may need this fact in the HW. You can just use it. You don't need to derive it every time. This is called Euler's formula (as are so many formulas from his time.)

Higher Order Systems

Consider the second order system:

$$Y'' = A_1 Y + A_2 Y'$$

This can be made into a first order system by setting

$$\begin{aligned} y_{n+1} &= y_1' \\ y_{n+2} &= y_2' \\ &\vdots \\ y_{2n} &= y_n' \end{aligned}$$

then letting $Y_1 = (y_1, \dots, y_n)^T$ and $Y_2 = (y_{n+1}, \dots, y_{2n})^T$;

$$Y_1' = 0Y_1 + IY_2$$
$$\text{and } Y_2' = A_1 Y_1 + A_2 Y_2.$$

These combine to give the $2n \times 2n$ system

$$\begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

If we know $Y_1(0) = Y(0) = Y_0$ and $Y_2(0) = Y'(0) = Y_0'$, then this becomes an IVP and has a unique solution.

Ex. Solve the system:

$$y_1'' = 2y_1 + y_2 + y_1' + y_2'$$
$$y_2'' = -5y_1 + 2y_2 + 5y_1' - y_2'$$

$$y_1(0) = y_2(0) = y_1'(0) = 4 \qquad y_2'(0) = -4$$

Putting $y_3 = y_1'$ and $y_4 = y_2'$ this yields

$$\begin{aligned} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= 2y_1 + y_2 + y_3 + y_4 \\ y_4' &= -5y_1 + 2y_2 + 5y_3 - y_4 \end{aligned}$$

or $Y' = AY$ w/ $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{pmatrix}$

A has e'values $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 3$, $\lambda_4 = -3$

$\underline{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $\underline{x}_2 = \begin{pmatrix} 1 \\ 5 \\ -1 \\ -5 \end{pmatrix}$, $\underline{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$, $\underline{x}_4 = \begin{pmatrix} 1 \\ -5 \\ -3 \\ 15 \end{pmatrix}$

The solution of the IVP is $\underline{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

so $Y = 2\,\underline{x}_1 e^{t} + \underline{x}_2 e^{-t} + \underline{x}_3 e^{3t}$

RE. Work out all of the details.

Diagonalization

We wish to decompose a matrix A into a product of the form $A = XDX^{-1}$, where D is diagonal, if possible.

Thm. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct e'values of an $n \times n$-matrix A w/ e'vectors $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_k$, then $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_k$ are linearly independent.

Pf. p. 307. Maybe a good problem to show all of the details?

Def'i. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be <u>diagonalizable</u> iff $\exists$ a non singular matrix X and a diagonal matrix D such that

$$A = XDX^{-1}$$

We say that X <u>diagonalizes</u> A.

Thm. An $n \times n$-matrix A is diagonalizable iff A has n linearly independent e'vectors.

Ex. Let $A = \begin{pmatrix} 2 & -3 \\ 2 & -5 \end{pmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = -4$

$\underline{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \underline{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$X = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \qquad X^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\text{so} \quad A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{pmatrix}$$

<u>RE.</u> Verify this.