Chapter 6: Eigenvalues

Def'n Let $A \in \mathbb{R}^{n \times n}$. A scalar λ is said to be an eigenvalue of A if there exists a nonzero vector $\underline{x} \in \mathbb{R}^n$ such that $A\underline{x} = \lambda \underline{x}$.
The vector $\underline{x}$ is called the eigenvector of A corresponding to λ.

Ex. Let $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ $\quad \underline{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$A\underline{x} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3\underline{x}$$

so $\underline{x}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 3$.

Notice: any scalar multiple of $\underline{x}$ also satisfies this, so eigenvectors are not unique.

It might be natural to take the unit vector however.

$$\underline{x} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The eigenvalue equation $A\underline{x} = \lambda \underline{x}$ can be rewritten in the form $A\underline{x} - \lambda\underline{x} = \underline{0}$, or

$$(A - \lambda I)\underline{x} = \underline{0}.$$

If λ is an eigenvalue, then $\det(A - \lambda I) = \underline{0}$, and $N(A - \lambda I)$ is called the eigenspace of λ.

If we expand the determinant, this becomes a polynomial in λ.

$$p(\lambda) = \det(A - \lambda I)$$

called the <u>characteristic polynomial</u> of A. The eigenvalues are the roots of this polynomial, provided their eigenspace is not null. (i.e., $\{0\}$).

<u>Thm.</u> Let $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$. The following are equivalent.

1. λ is an eigenvalue of A.
2. $(A-\lambda I)\underline{x} = \underline{0}$ has a nontrivial sol'n.
3. $N(A-\lambda I) \neq \{\underline{0}\}$.
4. $A-\lambda I$ is nonsingular
5. $\det(A-\lambda I) = 0$.

<u>Ex</u> $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$ Find the e'values, and e'spaces.

$$\det\begin{pmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) - 6$$
$$= -6 - \lambda + \lambda^2 - 6$$
$$= \lambda^2 - \lambda - 12$$
$$= (\lambda - 4)(\lambda + 3)$$

roots are $\lambda = 4$, $\lambda = -3$.

$\lambda = 4$: $A - 4I = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}$ $N(A-4I)$: $\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right)$

$$\rightarrow \left(\begin{array}{cc|c} 3 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} x_2 = \alpha \\ x_1 = 2\alpha \end{array}$$

$$N(A-4I) = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}.$$

So $\underline{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an e'vector for $\lambda = 4$.

$\lambda = -3$: $\quad A + 3I = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \quad \left(\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} x_2 = -3\alpha \\ x_1 = \alpha \end{array}$

So $N(A + 3I) = \text{span}\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$

and $\underline{x} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an e'vector for $\lambda = -3$.

__Ex.__ $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix} \qquad \lambda_1 = 0, \ \lambda_2 = \lambda_3 = 1$

$N(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$. Compute all of these!

$N(A - I) = \text{span}\left\{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

__Ex.__ $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad \begin{array}{l} \lambda_1 = 1 + 2i \\ \lambda_2 = 1 - 2i \end{array}$

$N(A - \lambda_1 I) = \text{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}$ and Compute!

$N(A - \lambda_2 I) = \text{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\}$.

There is more to do once we get comfortable w/ these values and the process of finding them.

Products and Sums of evalues

Let A be an $n \times n$ matrix w/ characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & & & \\ & a_{22}-\lambda & & \\ & & \ddots & \\ & & & a_{nn}-\lambda \end{vmatrix}$$

By arguments on pg. 292, we obtain that

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

$$p(0) = \det(A - 0\lambda) = \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

Therefore the determinant of A is the product of its eigenvalues!

It also follows that the trace of A

$$tr(A) = \Sigma a_{ii} = \Sigma \lambda_i$$

is the sum of the eigenvalues.

Recall that trace and det are independent of choice of basis, so these are the most natural representations!

Ex. $A = \begin{pmatrix} 5 & -18 \\ 1 & -1 \end{pmatrix}$ Find $\det(A)$ and $tr(A)$ using both methods. $\det(A) = 13$, $tr(A) = 4$.

Similar matrices.

In GP2 # 4 you will show that if $A \sim B$,

then $(A - \lambda I) \sim (B - \lambda I)$ for all λ.

therefore A and B have the same characteristic polynomial, hence the same eigenvalues.

This is as it must be since similar matrices are just different representations of the same linear operator.

pf. Good problem 4 b.

Ex. $T = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ $\quad S = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$

e'values of T are 2, 3. Put $A = S^{-1} T S$.
Show that e'values of A are also 2, 3.