Ch 5. Orthogonality

5.1 The scalar product in $\mathbb{R}^n$.

Let $\underline{x}, \underline{y} \in \mathbb{R}^n$, then

$$\langle \underline{x}, \underline{y} \rangle = \underline{x} \cdot \underline{y} := \underline{x}^T \underline{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Ex. $\underline{x} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ $\underline{y} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ $\underline{x}^T \underline{y} = 8$

The Euclidean length, or norm, of a vector $\underline{x}$ is given by

$$\| \underline{x} \| = \sqrt{\langle \underline{x}, \underline{x} \rangle} = \sqrt{x_1^2 + \cdots + x_n^2}$$

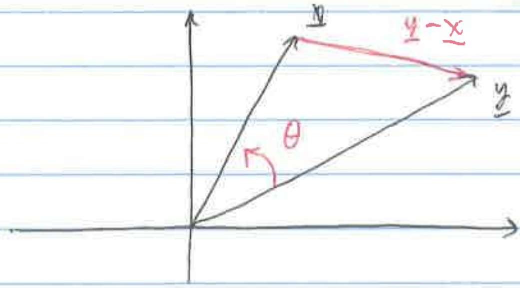
Defn. The distance between two points $\underline{x}, \underline{y} \in \mathbb{R}^n$ is given by $d(\underline{x}, \underline{y}) = \| \underline{x} - \underline{y} \|$.

$\underline{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\underline{y} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$

Ex. $d(\underline{x}, \underline{y})$ from last example is No.

Thm. If $\underline{x}, \underline{y} \in \mathbb{R}^n$ are nonzero, then the angle θ between them is defined by

$$\langle \underline{x}, \underline{y} \rangle = \| \underline{x} \| \| \underline{y} \| \cos \theta$$



Proof. By the law of cosines, we have

$$\|y - x\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos\theta$$

$$\Rightarrow \quad \|x\|\,\|y\|\cos\theta = \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x-y\|^2\right)$$

$$= \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \langle x-y, x-y\rangle\right)$$

$$= \frac{1}{2}\left(\langle x,x\rangle + \langle y,y\rangle - \left(\langle x,x\rangle + \langle y,y\rangle - 2\langle x,y\rangle\right)\right)$$

$$= \langle x, y\rangle \quad \square$$

Thus the angle between two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\,\|y\|}\right).$$

If u, v are unit vectors in the direction of x, y

$$u = \frac{x}{\|x\|} \quad \text{and} \quad v = \frac{y}{\|y\|},$$

then $\theta = \cos^{-1}(\langle u, v\rangle)$.

Ex. Find the angle between $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}-1\\7\end{pmatrix}$.

Get $\pi/4$

Thm <u>The Cauchy-Schwarz Inequality</u>

If $\underline{x}, \underline{y} \in \mathbb{R}^n$, then

$$|\langle \underline{x}, \underline{y} \rangle| \leq \|\underline{x}\| \|\underline{y}\|.$$

Proof.

$$|\langle \underline{x}, \underline{y} \rangle| = \big| \|\underline{x}\| \|\underline{y}\| \cos\theta \big|$$

$$= \|\underline{x}\| \|\underline{y}\| |\cos\theta| \qquad \text{but } |\cos\theta| \leq 1$$

$$\leq \|\underline{x}\| \|\underline{y}\|. \quad \square$$

Note. We get equality iff

$\theta = 0$ or π. This means the vectors are parallel.

If $\theta = \pm\frac{\pi}{2}$, then $\cos\theta = 0$ and we have

$\langle \underline{x}, \underline{y} \rangle = 0$.

Defn. The vectors $\underline{x}, \underline{y} \in \mathbb{R}^n$ are <u>orthogonal</u> iff $\langle \underline{x}, \underline{y} \rangle = 0$.

Ex. $\underline{0}$ is orth to everything.

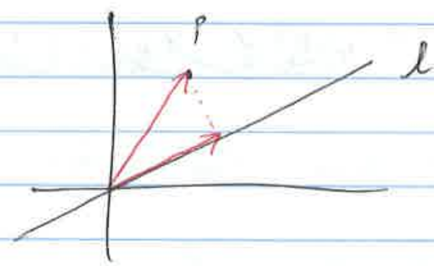
$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \perp \begin{pmatrix} -4 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Projections. $\quad proj_{\underline{v}}\, \underline{x} = \dfrac{\langle \underline{v}, \underline{x} \rangle}{\langle \underline{v}, \underline{v} \rangle}\, \underline{v}$

the length of this is the scalar projection.

well, no. $\dfrac{\langle \underline{v}, \underline{x} \rangle}{\sqrt{\langle \underline{v}, \underline{v} \rangle}} = \dfrac{\underline{v}^T \underline{x}}{\| \underline{v} \|}$ is the scalar proj. (could be negative)

Ex. $y = \frac{1}{3} x$ Find the pt on the line closest to $(1, 4)$.



$\underline{x} = \langle 1, 4 \rangle$

$\underline{v} = \langle 3, 1 \rangle$

$$proj_{\underline{v}}\, \underline{x} = \frac{\langle \underline{v}, \underline{x} \rangle}{\langle \underline{v}, \underline{v} \rangle}\, \underline{v}$$

$$= \frac{7}{10}\, \underline{v} = \left\langle \frac{21}{10}, \frac{7}{10} \right\rangle$$

$$= \langle 2.1, 0.7 \rangle$$

Normal vector $\underline{n}$ to a plane is s.t.

$$(\overrightarrow{P_0P_1})^T \underline{n} = \underline{0} \quad \text{for all } \overrightarrow{P_0P_1}$$

$\underline{p} = \langle x - x_0, y - y_0, z - z_0 \rangle \qquad \underline{n} = \langle a, b, c \rangle$

so $\underline{p}^T \underline{n} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Ex. Find the eqn of the plane through $(2,-1,3)$ w/ normal $\underline{n} = \langle 2,3,4 \rangle$.

by prev. remarks.

$$\pi: \; 2(x-2) + 3(y+1) + 4(z-3) = 0.$$

Ex. $P_1 = (1,1,2) \qquad P_2 = (2,3,3) \qquad P_3 = (3,-3,3)$

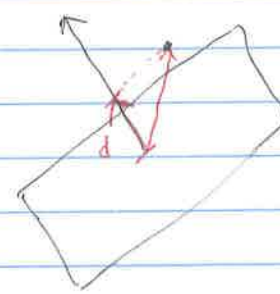
$$\underline{x} = \overrightarrow{P_1P_2} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \overrightarrow{P_1P_3}$$

$$\underline{n} = N = \underline{x} \times \underline{y} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$$

The plane is $6(x-1) + (y-1) - 8(z-2) = 0.$

Ex. Distance from pt to plane:

$$d = \frac{|\underline{v}^T \underline{n}|}{\|\underline{n}\|}$$



$p = (2,0,0) \qquad \pi: \; x + 2y + 2z = 0.$

$$d = \tfrac{2}{3}.$$