4.2. Matrix Representations

We touched on this in the last section.

For every matrix $A \in \mathbb{R}^{m \times n}$, there is a linear transformation $L_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$L_A(\underline{x}) = A\underline{x}.$$

Thm. Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a matrix $A \in \mathbb{R}^{m \times n}$ s.t.

$$L(\underline{x}) = A\underline{x} \quad \text{for all } \underline{x} \in \mathbb{R}^n.$$

The j-th column of A is given by

$$\underline{a}_j = L(\underline{e}_j) \qquad j = 1, 2, \ldots, n.$$

Proof. Let $A = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$ with $\underline{a}_j$ defined as above. If $\underline{x} = (x_1, \ldots, x_n)^T$, then

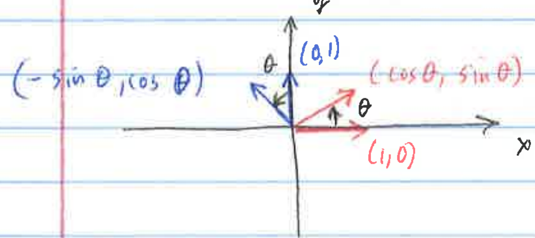
$$\begin{aligned} L(\underline{x}) &= L(x_1\underline{e}_1 + x_2\underline{e}_2 + \cdots + x_n\underline{e}_n) \\ &= x_1 L(\underline{e}_1) + x_2 L(\underline{e}_2) + \cdots + x_n L(\underline{e}_n) \\ &= x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n \\ &= (\underline{a}_1, \ldots, \underline{a}_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= A\underline{x}. \quad \square \end{aligned}$$

Ex. $L: \mathbb{R}^3 \to \mathbb{R}^2$ by $L(\underline{x}) = (x_1 + x_2, x_2 + x_3)^T$

$$A = \left(L(\underline{e}_1), L(\underline{e}_2), L(\underline{e}_3) \right) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

RE. Verify that $L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Ex. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ such that L rotates each vector by an angle θ in the counter-clockwise direction.



For the unit vectors $\underline{e}_1 = (1,0)$ and $\underline{e}_2 = (0,1)$, a rotation by θ amounts to finding the appropriate pt on the unit circle.

Let $\underline{x}$ be any vector in $\mathbb{R}^2$.

$$\text{The } L(\underline{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

since $\underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2$.

* This matrix should look familiar (exam 1).

Thm. Let $E = \{\underline{v}_1, \dots, \underline{v}_n\}$ and $F = \{\underline{w}_1, \dots, \underline{w}_m\}$ be ordered bases for two vector spaces V and W, resp. To each linear transformation $L: V \to W$ there is a matrix $A \in \mathbb{R}^{m \times n}$ satisfying

$$A[\underline{x}]_E = [L(\underline{x})]_F$$

for all $\underline{x} \in V$.

The matrix is given by $\underline{a}_j = [L(\underline{v}_j)]_F \quad j = 1, 2, \dots, n$

* This is essentially just an application of the previous theorem w/ change of bases.

Let $\underline{v} \in V$, $\underline{w} \in W$, $\underline{x} = [\underline{v}]_E$, and $\underline{y} = [\underline{w}]_F$.
The following diagram commutes:

$$\begin{tikzcd}
\underline{v} \in V \arrow[r, "L"] \arrow[r, bend left] \arrow[d, leftrightarrow] \arrow[d, bend right] & L(\underline{v}) = \underline{w} \in W \arrow[d, leftrightarrow] \\
\underline{x} \in \mathbb{R}^n \arrow[r, "A"'] \arrow[r, bend right] & A\underline{x} = \underline{y} \in \mathbb{R}^m \arrow[u, bend right]
\end{tikzcd}$$

In other words, following the red arrow or the blue arrows gives the same answer.

<u>Ex.</u> $L: \mathbb{R}^3 \to \mathbb{R}^2$ by $L(\underline{x}) = x_1 \underline{b}_1 + (x_2 + x_3)\underline{b}_2$ where
$\underline{b}_1 = (1,1)^T$ and $\underline{b}_2 = (-1,1)^T$.
Find $A \in \mathbb{R}^{2\times 3}$ that represents L.

$$L(\underline{e}_1) = 1\,\underline{b}_1 + 0\,\underline{b}_2 = \begin{pmatrix}1\\0\end{pmatrix}_b$$
$$L(\underline{e}_2) = 0\,\underline{b}_1 + 1\,\underline{b}_2 = \begin{pmatrix}0\\1\end{pmatrix}_b$$
$$L(\underline{e}_3) = 0\,\underline{b}_1 + 1\,\underline{b}_2 = \begin{pmatrix}0\\1\end{pmatrix}_b$$

so $A = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}$

<u>Ex.</u> $L: \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\alpha\underline{b}_1 + \beta\underline{b}_2) = (\alpha+\beta)\,\underline{b}_1 + 2\beta\,\underline{b}_2$

$$L(\underline{b}_1) = \underline{b}_1 = \begin{pmatrix}1\\0\end{pmatrix}_b$$
$$L(\underline{b}_2) = \underline{b}_1 + 2\,\underline{b}_2 = \begin{pmatrix}1\\2\end{pmatrix}_b$$

so $A = \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$

Ex. $D: P_3 \to P_2$ by $D(p) = p'$, w/ ordered bases $\{x^2, x, 1\}$ and $\{x, 1\}$.

$$D(x^2) = 2x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$
$$D(x) = 1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$D(1) = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Now consider a general element $p(x) = ax^2 + bx + c = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$D(p) = 2ax + b = \begin{pmatrix} 2a \\ b \end{pmatrix}$ as we know.

$$Ap = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ b \end{pmatrix}$$

so differentiation of polynomials is just multiplication by a matrix!

Thm. Let $E = \{\underline{u}_1, \dots, \underline{u}_n\}$ and $F = \{\underline{b}_1, \dots, \underline{b}_m\}$ be ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$. If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and A is the matrix representing L with respect to E and F, then

$$\underline{a}_j = B^{-1} L(\underline{u}_j) \qquad j = 1, \dots, n$$

where $B = (\underline{b}_1, \dots, \underline{b}_m)$.

(so B^{-1} is the change of basis matrix from $\underline{e}_i$ to $\underline{b}_i$.)

Cor. If A is the matrix representation of $L: \mathbb{R}^n \to \mathbb{R}^m$ wrt the bases E and F, then the RREF of $(B \mid L(\underline{u}_1) \cdots L(\underline{u}_n))$ is $(I \mid A)$, where B is the change of basis matrix $F \to I$. □

Ex. $L: \mathbb{R}^2 \to \mathbb{R}^3$ by $L(\underline{x}) = \begin{pmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$

let $\underline{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\underline{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\underline{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\underline{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$L(\underline{u}_1) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \qquad L(\underline{u}_2) = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{so} \quad B^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and $$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{pmatrix}}$$

RE. Verify that $L(\underline{u}_1) = -\underline{b}_1 + 4\underline{b}_2 - \underline{b}_3$ and
$L(\underline{u}_2) = -3\underline{b}_1 + 2\underline{b}_2 + 2\underline{b}_3$.

Homogeneous Coordinates

Embed $\mathbb{R}^2$ in $\mathbb{R}^3$ as $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$

If $L: \mathbb{R}^2 \to \mathbb{R}^2$ has matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then L can be extended to $L: \mathbb{R}^2 \to \mathbb{R}^3$ with matrix $A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a x_1 + b x_2 \\ c x_1 + d x_2 \end{pmatrix} \qquad A \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a x_1 + b x_2 \\ c x_1 + d x_2 \\ 1 \end{pmatrix}$$