Chapter 4: Linear Transformations

4.1. Definitions and Examples

Defn. A map $L: V \to W$ from a vector space V to a vector space W is called a linear transformation if

$$L(\underline{x} + \underline{y}) = L(\underline{x}) + L(\underline{y}) \quad \text{and}$$
$$L(\alpha \underline{x}) = \alpha L(\underline{x})$$

for all $\underline{x}, \underline{y} \in V$ and all $\alpha \in \mathbb{R}$.

If $V = W$, then $L: V \to V$ is called a linear operator on V.

Ex. Let L be the operator defined by $L(\underline{x}) = 3\underline{x}$, $\underline{x} \in \mathbb{R}^2$.

stretches by 3.

$$L(\alpha \underline{x}) = 3(\alpha \underline{x}) = \alpha(3\underline{x}) = \alpha L(\underline{x})$$
$$L(\underline{x} + \underline{y}) = 3(\underline{x} + \underline{y}) = 3\underline{x} + 3\underline{y} = L(\underline{x}) + L(\underline{y}).$$

Ex. $L(\underline{x}) = x_1 \underline{e}_1$, $\underline{x} \in \mathbb{R}^2$

projects to x_1-axis

Show that L is a linear operator.

$$L(\alpha \underline{x}) = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = (\alpha x_1)\underline{e}_1 = \alpha(x_1 \underline{e}_1) = \alpha L(\underline{x})$$

$$L(\underline{x} + \underline{y}) = (x_1 + y_1)\underline{e}_1 = x_1 \underline{e}_1 + y_1 \underline{e}_1 = L(\underline{x}) + L(\underline{y}).$$

Ex. $L(\underline{x}) = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$, $\underline{x} \in \mathbb{R}^2$.

reflects over x_1-axis

Show that L is a linear operator

Ex. $L(\underline{x}) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$. Rotates the vector by $\pi/2$ $(90°)$. counter clockwise.

* If we view $\mathbb{R}^2$ as the complex plane, this is multiplication by i.

Ex. $L : \mathbb{R}^2 \to \mathbb{R}^1 : \quad L(\underline{x}) = x_1 + x_2$

Ex. $d : \mathbb{R}^2 \to \mathbb{R}^1 : \quad d(\underline{x}) = \sqrt{x_1^2 + x_2^2}$

$$d(\alpha \underline{x}) = |\alpha| \, d(\underline{x}) \neq \alpha \, d(\underline{x}) \text{ in general}$$

so this is not a linear transformation.

Ex. $L(\underline{x}) = \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix}$ $\quad \underline{x} \in \mathbb{R}^2$ is linear

Notice that if $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

then $L(\underline{x}) = A \underline{x}$ for all $\underline{x}$

In general we can always associate linear transformations w/ matrices.

Given a matrix $A \in \mathbb{R}^{m \times n}$, define a lin. trans.

$$L_A(\underline{x}) := A \underline{x} \quad \text{from } \mathbb{R}^n \text{ to } \mathbb{R}^m.$$

Given a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, the matrix is given by

$$A_L = \Big(L(\underline{e}_1), L(\underline{e}_2), \ldots, L(\underline{e}_n) \Big)$$

Apply the lin. trans. to the std basis.

Some properties: $L: V \to W$ linear transformation.

1.) $L(\underline{0}_V) = \underline{0}_W$

2.) $L(\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \dots + \alpha_n \underline{v}_n) = \alpha_1 L(\underline{v}_1) + \alpha_2 L(\underline{v}_2) + \dots + \alpha_n L(\underline{v}_n)$

3.) $L(-\underline{v}) = -L(\underline{v})$ for all $\underline{v} \in V$.

RE. Prove all of these.

Ex. If V is any vector space, the identity linear trans. is

$$I(\underline{v}) = \underline{v} \quad \text{for all } \underline{v} \in V.$$

Ex. $L: C[a,b] \to \mathbb{R}$ by

$$L(f) = \int_a^b f(x)\,dx$$

$D: C^1[a,b] \to C[a,b]$ by

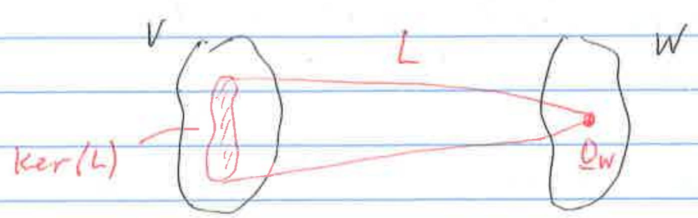
$$D(f) = f'$$

! RE Show that these are linear transformations!

Kernel and Image

Defn. Let $L: V \to W$ be a linear transformation. The *kernel* of L, denoted $\ker(L)$, is defined by

$$\ker(L) = \{ \underline{v} \in V \mid L(\underline{v}) = \underline{0}_W \}$$



Def'n. Let $L: V \to W$ be a linear transformation and S a subspace of V. The image of S, denoted by $L(S)$, is defined by

$$L(S) = \{ \underline{w} \in W \mid \underline{w} = L(\underline{v}) \text{ for some } \underline{v} \in S \}$$

The image of the entire space V is called the range of L.

Thm. If $L: V \to W$ is a linear transformation and S is a subspace of V, then

1.) $\ker(L)$ is a subspace of V
2.) $L(S)$ is a subspace of W

! RE Prove this! (cf. p. 173)

Ex. $L: \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\underline{x}) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

The kernel is spanned by elements of the form $\begin{pmatrix} 0 \\ \alpha \end{pmatrix}$, so it forms a 1-d subspace.

Ex. $L: \mathbb{R}^3 \to \mathbb{R}^2$ by $L(\underline{x}) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$

$$\ker(L) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Ex. $D: P_3 \to P_3$ by $D(p(x)) = p'(x)$

The kernel of D is all of P_1.
The image $D(P_3) = P_2$.

Justify these claims!