3.6. Row Space and Column Space

Defn. If $A \in \mathbb{R}^{m\times n}$, the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the row space of A. The subspace of $\mathbb{R}^m$ spanned by the columns of A is called the column space.

Ex. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad$ $\text{Row}(A) = \alpha \underline{e}_1 + \beta \underline{e}_2$
$\text{Col}(A) = \mathbb{R}^2.$

Thm. Two row-equivalent matrices have the same row space.

Proof. If B is row-eq. to A, then the row vectors of B are linear combinations of row vectors of A. Hence $\text{Row}(B) \subseteq \text{Row}(A)$.
Repeat the argument swapping A and B to get $\text{Row}(A) \subseteq \text{Row}(B)$.
Thus $\text{Row}(A) = \text{Row}(B)$. $\square$

Defn. The rank of a matrix A is the dimension of its row (equiv. column) space.

Ex. $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \text{rank}(A) = 2.$$

Thm. The Consistency Theorem for Linear Systems

A linear system $A\underline{x} = \underline{b}$ is consistent if and only if $\underline{b} \in \text{Col}(A)$.

Ex. If $\underline{b} = \underline{0}$, then this becomes

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + x_3 \underline{a}_3 + \dots + x_n \underline{a}_n = \underline{0}.$$

It follows that $A\underline{x} = \underline{0}$ will have only the trivial soln $\underline{x} = \underline{0}$ if and only if the columns of A are linearly independent.

Thm. Let $A \in \mathbb{R}^{m \times n}$. The linear system $A\underline{x} = \underline{b}$ is consistent for every $\underline{b} \in \mathbb{R}^m$ iff $\mathrm{Col}(A) = \mathbb{R}^m$; i.e., the columns of A span $\mathbb{R}^m$. The system $A\underline{x} = \underline{b}$ has at most one soln for every $\underline{b} \in \mathbb{R}^m$ if and only if the column vectors of A are linearly independent.

Cor. $A \in \mathbb{R}^{n \times n}$ is nonsingular if and only if the columns of A form a basis for $\mathbb{R}^n$.

The rank and the dimension of the null space always add up to the number of columns of the matrix. The dimension of the null space is called <u>nullity</u>.

Thm. <u>The Rank-Nullity Theorem</u>
If $A \in \mathbb{R}^{m \times n}$, then the rank of A and the nullity add up to n.

Proof. Let U be the RREF form of A. The system $A\underline{x} = \underline{0}$ is equivalent to the system $U\underline{x} = \underline{0}$. If A has rank r, then U will have r nonzero rows, and consequently the system $U\underline{x} = \underline{0}$ will have r lead variables and $n-r$ free variables. The dimension of $N(A)$ equals the number of free variables.

Ex. $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}$

Find bases for Row(A) and N(A). Verify that $N(A) = n - r$.

The RREF of A is

$$U = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $\{(1,2,0,3), (0,0,1,2)\}$ forms a basis of the row space. Thus $\text{rank}(A) = 2$.

The null space is obtained by solving $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{aligned} x_4 &= \alpha \\ x_2 &= \beta \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_3 &= -2\alpha \\ x_1 &= -2\beta - 3\alpha \end{aligned}$$

So the null space is spanned by $\left\{ \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Notice that the rank of $A = 2$ and nullity of $A = 2$.
$A \in \mathbb{R}^{3 \times 4}$.

Column Spaces

Thm. $A \in \mathbb{R}^{m \times n}$. The dimension of the row space of A equals the dimension of the column space.

Proof. If $A \in \mathbb{R}^{m \times n}$ has $\text{rank}(A) = r$, the row echelon form U of A will have r leading 1s. The columns of U corresponding to the leading 1s will be linearly independent.

They do <u>not</u> form a basis of the column space, however, since in general A and U will have different columns spaces. (It's ROW echelon form, not column echelon.) Let U_L be the matrix obtained from U by deleting all of the columns corresponding to free variables. Delete the same columns of A and denote the new matrix by A_L.

The columns of A_L form a basis for the column space of A.

<u>RE.</u> Finish the proof as on p. 158. □

<u>Ex.</u> $A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 15 & 5 \end{pmatrix} \xrightarrow{REF} \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Thus $\{\underline{a}_1, \underline{a}_2, \underline{a}_5\}$ form a basis of the column space.

<u>Ex.</u> Find the dimension of the subspace of $\mathbb{R}^4$ spanned by

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \quad \underline{x}_2 = \begin{pmatrix} 2 \\ 5 \\ -3 \\ 2 \end{pmatrix}, \quad \underline{x}_3 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 0 \end{pmatrix}, \quad \underline{x}_4 = \begin{pmatrix} 3 \\ 8 \\ -5 \\ 4 \end{pmatrix}$$

The subspace spanned is the column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{REF} \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first two columns $\underline{x}_1$ and $\underline{x}_2$ form a basis for the column space, so the vectors span a 2-dimensional subspace of $\mathbb{R}^4$.