3.5. Change of Basis

The standard basis for $\mathbb{R}^2$ is $\{\underline{e}_1, \underline{e}_2\}$ so that any vector $\underline{x} \in \mathbb{R}^2$ can be written as

$$\underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2$$

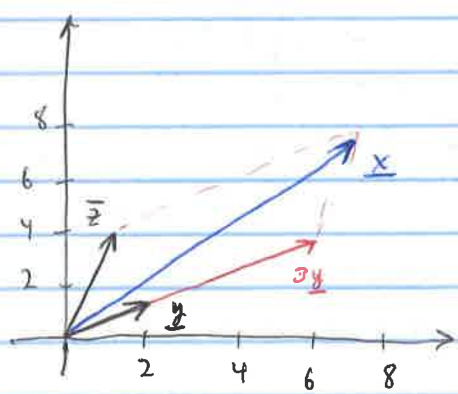
Suppose $\{\underline{y}, \underline{z}\}$ is another basis for $\mathbb{R}^2$. Then $\underline{x}$ can also be represented as

$$\underline{x} = \alpha \underline{y} + \beta \underline{z}$$

The ordered pairs $\underline{x} = (x_1, x_2)^T$ and $\underline{x} = (\alpha, \beta)^T$ are called the <u>coordinates</u> of $\underline{x}$ with respect to the <u>ordered</u> bases $\{\underline{e}_1, \underline{e}_2\}$ and $\{\underline{y}, \underline{z}\}$, respectively.

Ex. Let $\underline{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{z} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\underline{x} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$. Then $\underline{x} = 3\underline{y} + \underline{z}$.

The situation is represented geometrically as follows.



The coord's of $\underline{x}$ would be $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ in the (ordered) basis $\{\underline{y}, \underline{z}\}$.

We want to be able to switch between bases.

Change of Coordinates

Consider the standard basis $\{\underline{e}_1, \underline{e}_2\}$ of $\mathbb{R}^2$ and the ordered basis $\{\underline{u}_1, \underline{u}_2\}$ where

$$\underline{u}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \underline{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Problem 1. Given a vector $\underline{x} = c_1 \underline{u}_1 + c_2 \underline{u}_2$, find its coordinates wrt the standard basis.

We must write the basis vectors in terms of the standard basis vectors:

$$\underline{u}_1 = 3\underline{e}_1 + 2\underline{e}_2$$
$$\underline{u}_2 = \underline{e}_1 + \underline{e}_2$$

Then
$$\begin{aligned} c_1\underline{u}_1 + c_2\underline{u}_2 &= c_1(3\underline{e}_1 + 2\underline{e}_2) + c_2(\underline{e}_1 + \underline{e}_2) \\ &= (3c_1 + c_2)\underline{e}_1 + (2c_1 + c_2)\underline{e}_2 \\ &= \begin{pmatrix} 3c_1 + c_2 \\ 2c_1 + c_2 \end{pmatrix}, \text{ or} \end{aligned}$$

$$\underline{x} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Setting $U = (\underline{u}_1, \underline{u}_2) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, we see that $\underline{x} = U\underline{c}$.

U is called the transition matrix from the ordered basis $\{\underline{u}_1, \underline{u}_2\}$ to the standard basis $\{\underline{e}_1, \underline{e}_2\}$.

Problem 2. Given a vector $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, find its coord's wrt $\{\underline{u}_1, \underline{u}_2\}$.

Now we need the transition matrix from $\{\underline{e}_1, \underline{e}_2\}$ to $\{\underline{u}_1, \underline{u}_2\}$. Since U is nonsingular ($\underline{u}_1$ and $\underline{u}_2$ form a basis), then U^{-1} is exactly the matrix we need.

Then wrt $\{\underline{u}_1, \underline{u}_2\}$ the vector $\underline{x}$ is given by

$$\underline{c} = U^{-1}\underline{x}.$$

Ex. $\underline{u}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$. Find the coords of $\underline{x}$ wrt $\{\underline{u}_1, \underline{u}_2\}$.

$$U = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$\underline{c} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

so $\underline{x} = 3\underline{u}_1 - 2\underline{u}_2$.

Ex. $\underline{b}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\underline{b}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Find the transition matrix from $\{\underline{e}_1, \underline{e}_2\}$ to $\{\underline{b}_1, \underline{b}_2\}$, then find a coord expression for $\underline{x}$ wrt $\{\underline{b}_1, \underline{b}_2\}$.

$$B = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$B^{-1}\underline{x} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

So $\underline{x} = 7\underline{b}_1 + 3\underline{b}_2$.

Now consider the problem of changing from one basis $\{\underline{v}_1, \underline{v}_2\}$ to another $\{\underline{u}_1, \underline{u}_2\}$. Assume the coords for $\underline{x}$ are known in $\underline{V}$-basis. We want to convert to $\underline{u}$-basis.

Idea:

$$\begin{tikzcd} \{\underline{v}_1, \underline{v}_2\} \arrow[r, "V"] \arrow[rr, bend right, "U^{-1}V"'] & \{\underline{e}_1, \underline{e}_2\} \arrow[r, "U^{-1}"] & \{\underline{u}_1, \underline{u}_2\} \end{tikzcd}$$

Ex. $\underline{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $\underline{u}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Find the transition matrix $S = U^{-1}V$

$$V = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} \qquad U = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$S = U^{-1}V = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -4 & -5 \end{pmatrix}$$

Ex. Consider the bases $\{1, x, x^2\}$ and $\{1, 2x, 4x^2-2\}$. Find the transition matrix:

$$S = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

this one is $\{1, 2x, 4x^2-2\} \to \{1, x, x^2\}$.

The transition matrix back is given by

$$S^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

You should be able to see this "by inspection" at this point in the course. If not, keep practicing!