3.3. Linear Independence

Def'n. The vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ in a vector space V are said to be linearly independent if and only if

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_n \underline{v}_n = \underline{0}$$

implies that the scalars $c_1, c_2, \ldots, c_n$ must be 0.

Def'n. The vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ in a vector space V are linearly dependent if there exist scalars $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_n \underline{v}_n = \underline{0}.$$

Ex. The vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are linearly independent.

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 + c_2 = 0 \quad \text{and} \\ c_1 + 2c_2 = 0. \end{array}$$

Prop. (I) If $\underline{v}_1, \ldots, \underline{v}_n$ span a vector space V and one is a linear combination of the other $n-1$, then those $n-1$ span V.

(II) Given n vectors $\underline{v}_1, \ldots, \underline{v}_n$, it is possible to write one as a linear combination of the other $n-1$ iff they are linearly dependent.

Proof. (I) Suppose $\underline{v}_n = \beta_1 \underline{v}_1 + \cdots + \beta_{n-1} \underline{v}_{n-1}$

Let $\underline{x}$ be any element of V. Then

$$\begin{aligned}\underline{x} &= \alpha_1 \underline{v}_1 + \cdots + \alpha_{n-1} \underline{v}_{n-1} + \alpha_n \underline{v}_n \\ &= \alpha_1 \underline{v}_1 + \cdots + \alpha_{n-1} \underline{v}_{n-1} + \alpha_n (\beta_1 \underline{v}_1 + \cdots + \beta_{n-1} \underline{v}_{n-1}) \\ &= (\alpha_1 + \alpha_n \beta_1) \underline{v}_1 + \cdots + (\alpha_{n-1} + \alpha_n \beta_{n-1}) \underline{v}_{n-1}.\end{aligned}$$

(II) Spse $\underline{v}_n = \beta_1 \underline{v}_1 + \cdots + \beta_{n-1} \underline{v}_{n-1}$.

Subtracting $\underline{v}_n$ from both sides, we get

$$\underline{0} = \beta_1 \underline{v}_1 + \cdots + \beta_{n-1} \underline{v}_{n-1} - \underline{v}_n$$

Then $c_n = -1$, so $\underline{v}_1, \ldots, \underline{v}_n$ are lin. dep.

Conversely, if $\underline{v}_1, \ldots, \underline{v}_n$ are lin. dep., the rearrange to solve for one whose coefficient was nonzero. (Write out the details.) $\square$

Cor. If $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ is a minimal spanning set, then $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent. $\square$

A minimal spanning set for V is called a basis of V.

Ex. Let $\underline{x} = (1,2,3)^T$. The vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{x}$ are linearly dependent since

$$\underline{x} = \underline{e}_1 + 2\underline{e}_2 + 3\underline{e}_3$$

$$\Rightarrow \quad \underline{e}_1 + 2\underline{e}_2 + 3\underline{e}_3 - \underline{x} = \underline{0}.$$

Geometric Interpretation

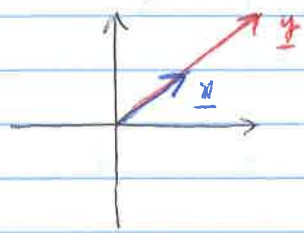
If $\underline{x}$ and $\underline{y}$ are linearly dependent in $\mathbb{R}^2$, then

$$c_1 \underline{x} + c_2 \underline{y} = \underline{0} \quad \text{for some } c_1, c_2 \text{ not both } 0.$$

Assuming $c_1 \neq 0$, this means that

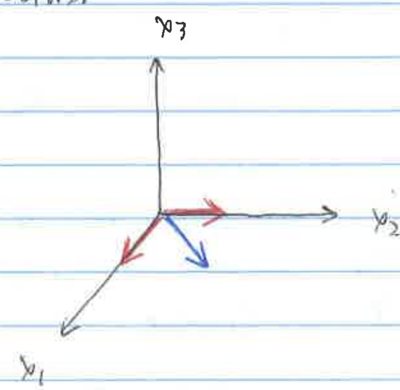
$$\underline{x} = \frac{-c_2}{c_1} \underline{y}$$

or that $\underline{x}$ is a constant multiple of $\underline{y}$. This means that $\underline{x}$ and $\underline{y}$ lie on the same "line".



So if a set of n vectors is linearly dependent, then at least one of the vectors lies in the linear subspace spanned by the other $n-1$ vectors.

Ex $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$



All three vectors live in the xy-plane (2-dim), so they must be lin. dep.

Any pair of two is linearly independent.

Ex. Which collections of vectors are linearly independent in $\mathbb{R}^3$?

a) $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$

b) $\{(1,0,1)^T, (0,1,0)^T\}$

c) $\{(1,2,4)^T, (2,1,3)^T, (4,-1,1)^T\}$

Sol'n. a) $c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Write this system as an augmented matrix:

$$\left(\begin{array}{ccc|c}1&1&1&0\\1&1&0&0\\1&0&0&0\end{array}\right) \Rightarrow c_1=c_2=c_3=0$$

So these vectors are lin. ind.

b) $\left(\begin{array}{cc|c}1&0&0\\0&1&0\\1&0&0\end{array}\right) \Rightarrow c_1=c_2=c_3=0.$ So yes lin. ind.

c) $\begin{pmatrix}1&2&4\\2&1&-1\\4&3&1\end{pmatrix}$ Take det: $1(4)-2(6)+4(2)$
$= 4-12+8 = 0$

so No.

Thm. Let $\underline{x}_1, \ldots, \underline{x}_n$ be n vectors in $\mathbb{R}^n$ and let $X=(\underline{x}_1, \ldots, \underline{x}_n)$. The vectors $\underline{x}_1, \ldots, \underline{x}_n$ are linearly dependent if and only if X is singular. ∎

Ex. $\begin{pmatrix}4\\2\\3\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}2\\-5\\3\end{pmatrix}$ are linearly dependent.

<u>Ex.</u> $\underline{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}$, $\underline{x}_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \\ -2 \end{pmatrix}$, $\underline{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 7 \\ 7 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 0 \\ 2 & 1 & 7 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \quad c_2 = -c_3$$

so lin. dep.

<u>Thm.</u> Let $\underline{v}_1, \dots, \underline{v}_n$ be vectors in V. A vector $\underline{v} \in \operatorname{Span}\{\underline{v}_1, \dots, \underline{v}_n\}$ can be written <u>uniquely</u> as a linear combination of $\underline{v}_1, \dots, \underline{v}_n$ if and only if $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent.

<u>Proof.</u> Let $\underline{v} \in \operatorname{Span}\{\underline{v}_1, \dots, \underline{v}_n\}$. Then $\underline{v}$ can be written as

$$\underline{v} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \dots + \alpha_n \underline{v}_n$$

Suppose $\underline{v}$ can also be written as

$$\underline{v} = \beta_1 \underline{v}_1 + \beta_2 \underline{v}_2 + \dots + \beta_n \underline{v}_n .$$

Subtracting:

$$(\alpha_1 - \beta_1)\underline{v}_1 + (\alpha_2 - \beta_2)\underline{v}_2 + \dots + (\alpha_n - \beta_n)\underline{v}_n = \underline{0}$$

If $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent, then

$$\alpha_1 = \beta_1, \ \alpha_2 = \beta_2, \dots, \alpha_n = \beta_n \quad \text{and the lin. comb. is unique.}$$

On the other hand, then

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n = \underline{0}$$

for some $c_1, \dots, c_n$ not all zero.

Now set $\beta_i = \alpha_i + c_i$. Then

$$\begin{aligned} \underline{v} = \underline{v} + \underline{0} &= \alpha_1 \underline{v}_1 + \cdots \alpha_n \underline{v}_n + c_1 \underline{v}_1 + \cdots + c_n \underline{v}_n \\ &= (\alpha_1 + c_1)\underline{v}_1 + (\alpha_2 + c_2)\underline{v}_2 + \cdots + (\alpha_n + c_n)\underline{v}_n \\ &= \beta_1 \underline{v}_1 + \cdots + \beta_n \underline{v}_n \\ &= \underline{v} \end{aligned}$$

and the linear combination is not unique. $\square$

<u>Vector Spaces of Functions</u>

We want to test whether polynomials in P_n are linearly independent.

A set of vectors $\{p_1, \ldots, p_n\} \subset P_n$ are <u>linearly</u> <u>independent</u> iff

$$c_1 p_1 + c_2 p_2 + \cdots + c_n p_n = 0$$

implies $c_1 = c_2 = \cdots = c_n = 0$.

<u>Ex.</u> $p_1(x) = x^2 - 2x + 3$
$p_2(x) = 2x^2 + x + 8$
$p_3(x) = x^2 + 8x + 7$

$$c_1 p_1 + c_2 p_2 + c_3 p_3 = c_1 x^2 - 2c_1 x + 3c_1 + 2c_2 x^2 + c_2 x + 8c_2 + c_3 x^2 + 8c_3 x + 7c_3 = 0$$

$$\Rightarrow (c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) = 0$$

$$\text{iff} \begin{cases} c_1 + 2c_2 + c_3 = 0 \\ -2c_1 + c_2 + 8c_3 = 0 \\ 3c_1 + 8c_2 + 7c_3 = 0 \end{cases}$$

This system has a unique sol'n iff the matrix is nonsingular:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{pmatrix}$$ but $\det A = 0$, so these vectors are <u>linearly</u> <u>dependent</u>.

<u>Notice</u>: By RE 3.1.16! we could have applied the isomorphism $P_3 \cong \mathbb{R}^3$ at the very beginning of this problem and saved ourselves some work.

<u>The Vector space $C^{(n-1)}[a,b]$</u> and the Wronskian!

Let $f_1, f_2, \ldots, f_n$ be n functions in $C^{n-1}[a,b]$. These vectors are linearly independent iff

$$c_1 f_1 + c_2 f_2 + \cdots c_n f_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0.$$

We showed in §3.2 that $C^{n-1} \subset C^{n-2} \subset \cdots \subset C^2 \subset C^1 \subset C$. Therefore these functions must also satisfy:

$$\begin{aligned} c_1 f_1' + c_2 f_2' + \cdots + c_n f_n' &= 0 \\ c_1 f_1'' + c_2 f_2'' + \cdots + c_n f_n'' &= 0 \\ \vdots \qquad \vdots \qquad & \vdots \\ c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \cdots + c_n f_n^{(n-1)} &= 0 \end{aligned}$$

In other words, for every $x \in [a,b]$ the matrix eqn

$$\begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

will have only the trivial sol'n iff $f_1, \ldots, f_n$ are linearly independent.

Def'n. Let $f_1, \dots, f_n \in C^{n-1}[a,b]$. The function

$$W[f_1, \dots, f_n](x) = \left| \begin{pmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix} \right|$$

is called the <u>Wronskian</u> of $f_1, \dots, f_n$. It is defined on $[a,b]$.

Thm. Let $f_1, \dots, f_n \in C^{n-1}[a,b]$. If there exists a point $x_0 \in [a,b]$ such that $W[f_1, \dots, f_n](x_0) \neq 0$, then $f_1, \dots, f_n$ are linearly independent. This is a one way only theorem!

Proof. If $f_1, \dots, f_n$ were linearly dependent, then the coeff. matrix would be singular for every $x \in [a,b]$. Hence $W[f_1, \dots, f_n](x)$ would vanish on all of $[a,b]$. $\square$

Fact. If $f_1, \dots, f_n$ are linearly independent in $C^{n-1}[a,b]$ then they will also be linearly independent in $C^{n-2}, C^{n-3}, \dots, C^2, C^1$, and C.

RE. Prove it!

Ex. Show that e^x and e^{-x} are linearly independent in $C(-\infty, \infty) = C(\mathbb{R})$.

$$W[e^x, e^{-x}] = \left| \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \right| = -e^x e^{-x} - e^x e^{-x} = -2 \neq 0.$$

Ex. $$W[x^2, x|x|] = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 2x^2|x| - 2x^2|x| = 0 \quad \text{for all } x.$$

~~So x^2 and $x|x|$ are linearly dependent.~~ No !!

So the theorem doesn't tell us any information.
Instead, suppose that

$$c_1 x^2 + c_2 x|x| = 0$$

for all $x \in [-1,1]$. In particular, for $x=1$ and $x=-1$ this yields

$$c_1 + c_2 = 0 \quad \text{and}$$
$$c_1 - c_2 = 0$$

This implies that $c_1 = c_2 = 0$.

Therefore x^2 and $x|x|$ are linearly independent even though $W[x^2, x|x|] \equiv 0$. This shows that the converse of the theorem cannot hold.

<u>Ex.</u> Show that $1, x, x^2, x^3$ are linearly independent in $C(-\infty, \infty)$.

$$W[1, x, x^2, x^3] = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 1 \cdot 1 \cdot 2 \cdot 6 = 12 \neq 0.$$