Chapter 3: Vector Spaces

3.1. Definitions and Examples

Review the vector spaces $\mathbb{V}^n = \mathbb{R}^n = \mathbb{E}^n$ of calc 2, where you thought of a vector as a directed line segment (or arrow). Recall that only the length and direction of the vector were necessary to define the vector. It didn't matter where it was positioned.

RE. Review the scalar multiplication and vector addition from Calc II, and the geometric interpretation(s).

* We now want to define an abstract vector space as an algebraic object (or structure).

Def'n. ~~Let $(V,+,\cdot)$ be a set together w/ two operations:~~
~~$+$ = addition of two elements~~
Let V be a set on which addition and scalar multiplication are defined, and suppose that V is closed under these operations:

Closure properties
- C1. If $\underline{x} \in V$ and $\alpha \in \mathbb{R}$, then $\alpha \underline{x} \in V$.
- C2. If $\underline{x}, \underline{y} \in V$, then $\underline{x} + \underline{y} \in V$.

The set $(V,+,\cdot)$ is called a vector space if and only if:
- A1. $\underline{x} + \underline{y} = \underline{y} + \underline{x}$ for all $\underline{x}, \underline{y} \in V$
- A2. $(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z})$ $\forall\, \underline{x}, \underline{y}, \underline{z} \in V$
- A3. There exists an element $\underline{0} \in V$ s.t. $\underline{x} + \underline{0} = \underline{x}$ $\forall\, \underline{x} \in V$.
- A4. For every $\underline{x} \in V$, there exists an element $-\underline{x}$ in V s.t $\underline{x} + (-\underline{x}) = \underline{0}$.
- A5. $\alpha(\underline{x} + \underline{y}) = \alpha \underline{x} + \alpha \underline{y}$ for all $\alpha \in \mathbb{R}$, and all $\underline{x}, \underline{y} \in V$.
- A6. $(\alpha + \beta)\underline{x} = \alpha \underline{x} + \beta \underline{x}$ for all $\alpha, \beta \in \mathbb{R}$ and all $\underline{x} \in V$.
- A7. $(\alpha\beta)\underline{x} = \alpha(\beta \underline{x})$ for all $\alpha, \beta \in \mathbb{R}$ and all $\underline{x} \in V$
- A8. $1 \cdot \underline{x} = \underline{x}$ for all $\underline{x} \in V$.

Are all satisfied.

To see that the closure properties are necessary consider the following example.

Ex. Let $W = \{(a,1) \mid a \in \mathbb{R}\}$
$(3,1)$ and $(5,1)$ are in W, but $(3,1)+(5,1)=(8,2)$ is not in W.

This means the operation "+" is not really defined on W.
Similarly for $\cdot$.

Ex. Let $U = \{(a,a) \mid a \in \mathbb{R}\}$
Verify that U satisfies the axioms of a vector space.

C1. $(a,a)+(b,b) = (a+b, a+b)$
C2. $k(a,a) = (ka, ka)$
... etc.

The rest follow easily as well.
A3. $\underline{0} = (0,0)$ is in U. This property is necessary for any vector space. (obviously!)

* Now for some more abstract vector spaces:

Ex. $C[a,b]$

The space $C[a,b]$ is the set of all real-valued functions that are defined and continuous on the interval $[a,b]$.
The sum of two functions in $C[a,b]$ is defined point-wise,
$f, g \in C[a,b]$

$$(f+g)(x) = f(x) + g(x)$$

And scalar multiplication is also defined point-wise by

$$(\alpha f)(x) = \alpha \cdot f(x).$$

RE. Check all of the axioms!

A1. $f+g = g+f$ since $(f+g)(x) = f(x)+g(x)$
$$= g(x) + f(x)$$
$$= (g+f)(x) \quad \square$$

A3. $\underline{0} = 0(x) = 0$ for all $x \in [a,b)$, the zero function.

A4. $(-f)(x) = -f(x)$, and $f + (-f) = (f-f) = 0$.

Check the rest! * Note: $C[a,b]$ is an infinite-dimensional space.

Ex. The vector space P_n.

Let P_n denote the set of all polynomials of degree less than n. Define $p+q$ and αp pointwise just as in $C[a,b]$.

In this case, the zero vector is the zero polynomial

$$0(x) = \underset{\substack{\uparrow \\ \text{in the book}}}{z(x)} = 0x^{n-1} + 0x^{n-2} + \cdots + 0x + 0.$$

[recall that a polynomial looks like $p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$.]

There are some properties that are consequences of the axioms.

Thm. Let V be a vector space and $\underline{x} \in V$. Then

1.) $0\underline{x} = \underline{0}$
2.) if $\underline{x} + \underline{y} = \underline{0}$, then $\underline{x} = -\underline{y}$.
3.) $(-1)\underline{x} = -\underline{x}$

Proof. 1.) $\underline{x} = 1\underline{x} = (1+0)\underline{x} = 1\underline{x} + 0\underline{x} = \underline{x} + 0\underline{x}$

so

$-\underline{x} + \underline{x} = -\underline{x} + (\underline{x} + 0\underline{x}) = (-\underline{x} + \underline{x}) + 0\underline{x}$

and $\underline{0} = \underline{0} + 0\underline{x} = 0\underline{x}$ $\square$

2.) Suppose $\underline{x} + \underline{y} = \underline{0}$. Then

$-\underline{x} = -\underline{x} + \underline{0} = -\underline{x} + (\underline{x} + \underline{y}) = (-\underline{x} + \underline{x}) + \underline{y} = \underline{0} + \underline{y} = \underline{y}$

$\square$

3.) $\underline{0} = 0\underline{x} = (1 + (-1))\underline{x} = 1\underline{x} + (-1)\underline{x}$

Thus $\underline{x} + (-1)\underline{x} = \underline{0}$ and by Part 2.)

$(-1)\underline{x} = -\underline{x}$. $\square$

Let's look at some examples.

RE. 3.1.14 $\mathbb{Z}$ the set of all integers w/ usual +.

Define scalar mult. by

$$\alpha \circ k = [\![\alpha]\!] \cdot k \quad \text{for all } \alpha \in \mathbb{R},\ k \in \mathbb{Z}.$$

Show that $\mathbb{Z}$ is not a vector space. Which axioms fail?

A6. let $\alpha = 0.75$ $\quad \beta = 0.5$ $\quad \alpha + \beta = 1.25$

$[\![\alpha]\!] = 0$ $\quad [\![\beta]\!] = 0$ $\quad [\![\alpha+\beta]\!] = 1$

so $(\alpha+\beta)\underline{x} \neq \alpha\underline{x} + \beta\underline{x}$!

A7. let $\alpha = 4$ $\quad \beta = 0.5$ $\quad \alpha\beta = 2$

$[\![\alpha]\!] = 4$ $\quad [\![\beta]\!] = 0$ $\quad [\![\alpha\beta]\!] = 2$

$4 \cdot 0 \neq 2$!

RE 3.1.15 Let $S =$ the set of all infinite sequences of real numbers, with

$$\alpha(a_n) = (\alpha a_n)$$

and $$(a_n) + (b_n) = (a_n + b_n).$$

Show that S is a vector space.

RE 3.1.16 We can define a one-to-one correspondence between the elements of P_n and $\mathbb{R}^n$ by

$$p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$$

$$\updownarrow \cong$$

$$\underline{a} = (a_1, a_2, \ldots, a_n)^T$$

Show that if $p \leftrightarrow \underline{a}$ and $q \leftrightarrow \underline{b}$, then

a) $\alpha p \leftrightarrow \alpha \underline{a}$ for all $\alpha \in \mathbb{R}$ and

b) $p + q \leftrightarrow \underline{a} + \underline{b}$

These two properties mean that P_n and $\mathbb{R}^n$ are <u>isomorphic</u>.

RE 3.1.9 Let V be a vector space and $\underline{x} \in V$. Show that

a.) $\beta \underline{0} = \underline{0}$ for all $\beta \in \mathbb{R}$ and

b.) if $\alpha \underline{x} = \underline{0}$, then either $\alpha = 0$ or $\underline{x} = \underline{0}$.

RE 3.1.7! Show that the $\underline{0}$ element of a vector space is <u>unique</u>.

Suppose ~~that~~ there is another element $\underline{y} \in V$ s.t $\underline{x} + \underline{y} = \underline{x}$ for all $\underline{x} \in V$. Then, in particular, this holds for $\underline{x} = \underline{0}$ and we get

$$\underline{0} + \underline{y} = \underline{0}$$

but $$\underline{0} + \underline{y} = \underline{y}$$ by definition of $\underline{0}$.

Thus $\underline{y} = \underline{0}$. $\square$

RE 3.1.8! Let V be a vector space and $\underline{x}, \underline{y}, \underline{z} \in V$. Prove that if $\underline{x} + \underline{y} = \underline{x} + \underline{z}$, then $\underline{y} = \underline{z}$.

(Cancellation law.)

Proof. If $\underline{x} + \underline{y} = \underline{x} + \underline{z}$, then $-\underline{x} + (\underline{x} + \underline{y}) = -\underline{x} + (\underline{x} + \underline{z})$

$$\Rightarrow (-\underline{x} + \underline{x}) + \underline{y} = (-\underline{x} + \underline{x}) + \underline{z}$$

$$\Rightarrow \underline{0} + \underline{y} = \underline{0} + \underline{z}$$

$$\Rightarrow \underline{y} = \underline{z} \qquad \Box$$