2.3. Additional Topics: Adjoints and Cramer's Rule

Let $A \in \mathbb{R}^{n\times n}$. We can define a new $n\times n$ matrix called the adjoint of A by:

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & \vdots \\ \vdots & & & \\ A_{1n} & \cdots & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} is the $(i,j)^{th}$ cofactor of A, * Notice the transpose!

If A is nonsingular, then $\det(A) \neq 0$ and

$$A\left(\frac{1}{\det(A)} \operatorname{adj} A\right) = I.$$

RE. Verify this!

Thus, we obtain yet another way to find the inverse of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) \quad ; \quad \det(A) \neq 0.$$

Ex. We've seen this multiple times for $\mathbb{R}^{2\times 2}$.

Ex. Compute $\operatorname{adj} A$ and A^{-1} for $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

$$\operatorname{adj} A = \begin{pmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 2/5 & 1/5 & -2/5 \\ -7/5 & 4/5 & 2/5 \\ 4/5 & -3/5 & 1/5 \end{pmatrix}.$$

Cramer's Rule

Thm. (Cramer) $A \in \mathbb{R}^{n\times n}$ non singular, $\underline{b} \in \mathbb{R}^{n\times n}$. Let A_i be the matrix obtained by replacing the i^{th} column of A by $\underline{b}$.

If $\underline{x}$ is the unique solution of $A\underline{x} = \underline{b}$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \text{for } i = 1, \dots, n.$$

Proof. $$\underline{x} = A^{-1}\underline{b} = \frac{1}{\det(A)}(\operatorname{adj} A)\,\underline{b}$$

$$\det(A)\cdot x_i = \overrightarrow{(\operatorname{adj} A)_i}\,\underline{b} = b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}$$

$$= \det(A_i)$$

$$\Rightarrow \quad x_i = \frac{\det(A_i)}{\det(A)} \quad \square$$

Ex. Use Cramer's rule to solve:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 3x_1 - x_2 &= 1 \end{aligned}$$

$$\left.\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \xrightarrow{\det} -7 \\ A_1 &= \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \xrightarrow{\det} -5 \\ A_2 &= \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \xrightarrow{\det} -8 \end{aligned}\right\} \quad (x_1, x_2) = \left(\frac{5}{7}, \frac{8}{7}\right)$$

Ex. If A is singular, what can you say about the product $A\,(\mathrm{adj}\,A)$?

Ex. Show that if $\det(A) = 1$, then $\mathrm{adj}(\mathrm{adj}\,A) = A$.

Ex. If A is nonsingular, then $(\mathrm{adj}\,A)$ is non singular and $(\mathrm{adj}\,A)^{-1} = \mathrm{adj}\,(A^{-1})$.

Ex. A matrix is called self-adjoint iff $A = (\mathrm{adj}\,A)$. Can you think of an example? Can you think of another example?

Ex. If $A\underline{x} = \underline{y}$, what can you say about $(\mathrm{adj}\,A)\,\underline{y}$? What if $\det(A) = 1$?