2.2. Properties of Determinants

Lemma. $A \in \mathbb{R}^{n\times n}$. Let A_{jk} denote the cofactor of a_{jk}, $k=1,\dots,n$. Then,

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases}$$

Proof. If $i=j$, this is the cofactor expansion definition of $\det(A)$.

If $i\neq j$, let A^* be the matrix obtained by replacing the j^{th} row of A by the i^{th} row of A.

$$A^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i^{th}\text{ row} \\ \\ \leftarrow j^{th}\text{ row} \\ \\ \\ \end{array}$$

Since two rows of A^* are the same, $\det(A^*)=0$. The cofactor expansion of $\det(A^*)$ along row j is exactly the equation in the Lemma. [Verify this!]. □

Now, let's examine the result of ~~interchanging~~ performing row ops on a matrix before taking the determinant.

<u>Row Op I:</u> Interchange two rows.

<u>Ex.</u> $A\in\mathbb{R}^{2\times 2}$, $E=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. If $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$, then

$EA=\begin{pmatrix}a_{21}&a_{22}\\a_{11}&a_{12}\end{pmatrix}$ and $\det(EA)=a_{21}a_{12}-a_{11}a_{22}=-\det(A)$.

In general, suppose $A \in \mathbb{R}^{n \times n}$ and E is a type I elementary matrix.

Then $\det(EA) = -\det(A)$.

In particular, $\det(E) = \det(EI) = -\det(I) = -1$.

For any elementary matrix of type I, then

$$\det(EA) = -\det(A) = \det(E)\det(A).$$

* i.e., for ~~type I elem matrices~~, det distributes over multiplication by a type I elem. matrix.

! RE. Let E_{13} be the elementary matrix that interchanges rows 1 and 3 of a 3×3 matrix. Write out the details to show that:

1.) $\det(E_{13}) = -1$

2.) $\det(E_{13} A) = -\det(A)$ for any $A \in \mathbb{R}^{3 \times 3}$.

Row Operation II: Multiply a row by a nonzero constant.

Let E be the elementary matrix that multiplies a row by $\alpha \neq 0$, and consider the matrix EA.
Then the cofactor expansion of $\det(EA)$ is along the row multiplied by α is:

$$\det(EA) = \alpha a_{i1} A_{i1} + \alpha a_{i2} A_{i2} + \cdots + \alpha a_{in} A_{in} = \alpha \det(A).$$

In particular,

$$\det(E) = \det(EI) = \alpha \det(I) = \alpha, \text{ so}$$

$\det(EA) = \alpha \det A = (\det E)(\det A)$ for all E of type 2 and all $A \in \mathbb{R}^{n \times n}$.

Row Op III: Add a multiple of one row to another.

Let E be a type III elem. matrix; i.e., one obtained by performing a row op $cR_i + R_j \to R_j$ to the Id. matrix, recall that type III matrices always have 1s along the diagonal.
Thus $\det(E) = 1$.

RE. Verify this!

We will show that $\det(EA) = \det(A) = \det(E)\det(A)$.

Suppose $\det(EA)$ is expanded along the j^{th} row (the one that E has an "extra entry" in). Then

$$\det(EA) = (a_{j1} + ca_{i1})A_{j1} + (a_{j2} + ca_{i2})A_{j2} + \dots + (a_{jn} + ca_{in})A_{jn}$$

$$= (a_{j1}A_{j1} + a_{j2}A_{j2} + \dots + a_{jn}A_{j2}) + c\underbrace{(a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn})}_{=0 \text{ by the lemma!}}$$

$$= \det(A)$$

$$= \det(E)\det(A).$$

* Thus, we have shown that $\det(EA) = \det(E)\det(A)$ for all elementary matrices E and all matrices A, and

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is type I} \\ \alpha \neq 0 & \text{if } E \text{ is type II} \\ 1 & \text{if } E \text{ is type III} \end{cases}.$$

RE. Show that $\det(AE) = \det(A)\det(E)$.
Hint: Use transposes and a property we deduced in §2.1.

RE. Show that if one row or column of a matrix is a multiple of another, then the determinant must equal 0.

Thm. A matrix $A \in \mathbb{R}^{n \times n}$ is singular if and only if $\det(A) = 0$.

Proof. Every square matrix can be reduced to REF by a finite number of multiplications by elem. matrices:

$$U = E_k E_{k-1} \cdots E_2 E_1 A$$

where U is upper triangular. Then

$$\det(U) = \det(E_k)\det(E_{k-1}) \cdots \det(E_2)\det(E_1)\det(A)$$

and $\det(E_i) \neq 0$ for all i. Thus $\det(U) = 0$ if and only if $\det(A) = 0$.

If A is singular, the U has a row of all zeros and $\det(U) = 0 \Rightarrow \det(A) = 0$.

If A is nonsingular, then we may assume that $U = I$ (why!?), whence $\det(U) = 1$ and $\det(A) \neq 0$. $\square$

RE. Every nonsingular matrix can be decomposed as a product $A = LU$, where $\det(L) = 1$. U is upper triangular. Therefore the determinant of A is the product of the diagonal entries of U. Work out the details.

RE. Let $A \in \mathbb{R}^{n \times n}$, and $\alpha \in \mathbb{R}$. Show that

$$\det(\alpha A) = \alpha^n \det(A).$$

Thm. Let $A, B \in \mathbb{R}^{n \times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof. Suppose A is singular. Then by Thm 1.5.2, AB is also singular for all B.

RE. Verify this! [This is Ex. 18, §1.5]

Thus $\det(AB) = 0 = \det(A)\det(B)$ since $\det(A) = 0$.

Now suppose A is nonsingular. Then A is row-eq. to I (Thm. 1.5.2) and I "is" an elementary matrix, so $A = E_k E_{k-1} \cdots E_2 E_1$ is a product of elem. matrices.

Thus,

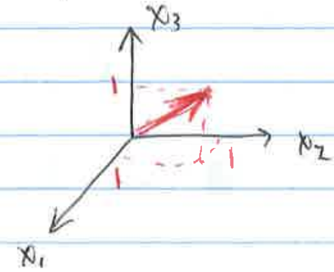
$$\begin{aligned} \det(AB) &= \det(E_k \cdots E_1 B) \\ &= \det(E_k) \cdots \det(E_1)\det(B) \\ &= \det(A)\det(B). \end{aligned}$$

□

Geometric Interpretation of the determinant in $\mathbb{R}^3$.

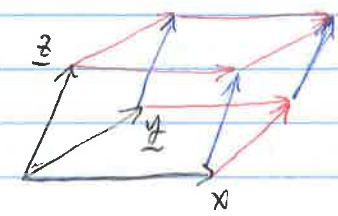
Let $\underline{x}, \underline{y}, \underline{z}$ be vectors in $\mathbb{R}^3$. From Calc II, these can be thought of as "arrows" emanating from the origin w/ the coordinates of the tip arrow coming from the ordered triple that is "our" 3-vector.

e.g., $\underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ corresponds to the arrow



in the first octant.

Three vectors in $\mathbb{R}^3$ "span" a parallelepiped:



(pretend these are all parallel) (well, the ones that <u>should</u> be.)

Suppose we form a 3×3 matrix by making $\underline{x}, \underline{y}$, and $\underline{z}$ the columns of the matrix.

$$A = (\underline{x}, \underline{y}, \underline{z})$$

Then the volume of the parallelepiped is given by

$$\text{Volume} = |\det(A)|.$$

Now let $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

These form the <u>standard basis</u> for $\mathbb{R}^3$.

If $(\underline{x}, \underline{y}, \underline{z})$ have the same orientation as $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ using the <u>Right Hand Rule</u> (Calc II or physics),

then $\det A > 0$. If the orientation is opposite, then $\det A < 0$.

RE. What is the effect of changing two adjacent columns of A? i.e., $(\underline{x}, \underline{y}, \underline{z}) \longmapsto (\underline{y}, \underline{x}, \underline{z})$?
Explain geometrically, what is the effect on the determinant? What elementary matrix achieves this switch?

RE. Now interchange two non adjacent entries.
$(\underline{x}, \underline{y}, \underline{z}) \mapsto (\underline{z}, \underline{y}, \underline{x})$.
Explain geometrically and find the elementary matrix that achieves this switch.

! RE. Show that every type I elementary matrix can be realized as the product of an odd number of type I matrices that interchange adjacent rows (or columns). (Prove this in $\mathbb{R}^{n\times n}$, not just $\mathbb{R}^{3\times 3}$.)

* We can't think in any dimensions higher than 3 since we can't see in them, but this geometric idea extends to higher dimensions as well.

$|\det(\ )|$ is the "volume form" in $\mathbb{R}^n$, and

sign $(\det(\ ))$ gives the orientation, where

$$\text{sign } X = \frac{X}{|X|}, \quad X \in \mathbb{R}.$$

Obviously, this only makes sense for nonsingular matrices.

Ex. Suppose $\det(A) = 0$ where $A = (\underline{x}, \underline{y}, \underline{z})$. What can you say about $\underline{x}$, $\underline{y}$, and $\underline{z}$?

Turn this around, if $\underline{x}, \underline{y}$, and $\underline{z}$ are <u>linearly dependent</u>, i.e., if they all lie in the same line or plane, then $\det(A) = 0$.

RE. Draw pictures of the possible ~~[illegible]~~ "alignments" of $\underline{x}, \underline{y}$, and $\underline{z}$. What can you say about the volume form and orientation form in each case?