1.6: Partitioned Matrices

It's sometimes useful to regard a single matrix as being composed of smaller submatrices (or blocks).

We've already seen this: $A = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) = \begin{pmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{pmatrix}$

RE. read pages 68 and 69.

Block Multiplication

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times r}$. There are at least 4 ways to carry out the mult. AB by partitioning A and or B.

Case 1. let $B = (B_1 \quad B_2)$, $B_1 \in \mathbb{R}^{n \times t}$, $B_2 \in \mathbb{R}^{n \times (r-t)}$

Then $\boxed{AB = (AB_1 \quad AB_2)}$

Case 2. Let $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, $A_1 \in \mathbb{R}^{k \times n}$, $A_2 \in \mathbb{R}^{(m-k) \times n}$

Then $\boxed{AB = \begin{pmatrix} A_1 B \\ A_2 B \end{pmatrix}}$.

Case 3. ~~let~~ $\cancel{A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}}$, $\cancel{B = (B_1, B_2)}$,

$A = (A_1 \quad A_2)$ $\quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ $\quad A_1 \in \mathbb{R}^{m \times s}$, $A_2 \in \mathbb{R}^{m \times (n-s)}$

$B_1 \in \mathbb{R}^{s \times r}$, $B_2 \in \mathbb{R}^{(n-s) \times r}$

so, $\boxed{AB = A_1 B_1 + A_2 B_2}$

Case 4.

$$A = \begin{array}{c} \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \begin{array}{l} k \\ m-k \end{array} \\ \begin{array}{cc} s & n-s \end{array} \end{array} \qquad B = \begin{array}{c} \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right) \begin{array}{l} s \\ n-s \end{array} \\ \begin{array}{cc} t & r-t \end{array} \end{array}$$

Now let $A_1 = \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} \in \mathbb{R}^{m \times s}$, $A_2 = \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} \in \mathbb{R}^{m \times (n-s)}$

$$B_1 = \begin{pmatrix} B_{11} & B_{12} \end{pmatrix} \in \mathbb{R}^{s \times r}, \quad B_2 = \begin{pmatrix} B_{21} & B_{22} \end{pmatrix} \in \mathbb{R}^{(n-s) \times r}$$

So $AB = A_1 B_1 + A_2 B_2$

$$= \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \end{pmatrix} + \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} \begin{pmatrix} B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} B_{11} & A_{11} B_{12} \\ A_{21} B_{11} & A_{21} B_{12} \end{pmatrix} + \begin{pmatrix} A_{12} B_{21} & A_{12} B_{22} \\ A_{22} B_{21} & A_{22} B_{22} \end{pmatrix}$$

$$= \begin{array}{c} \left(\begin{array}{c|c} A_{11} B_{11} + A_{12} B_{21} & A_{11} B_{12} + A_{12} B_{22} \\ \hline A_{21} B_{11} + A_{22} B_{21} & A_{21} B_{12} + A_{22} B_{22} \end{array}\right) \begin{array}{l} k \\ m-k \end{array} \\ \begin{array}{cc} t & r-t \end{array} \end{array}$$

Ex. $A = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 \end{array}\right) \qquad B = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right)$

Find AB w/ both partitions of A:

get $AB = \begin{pmatrix} 8 & 6 & 4 & 5 \\ 10 & 9 & 6 & 7 \\ 18 & 15 & 10 & 12 \end{pmatrix}$

Ex. Let $A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$ $A_{11}, A_{22} \in \mathbb{R}^{k \times k}, \mathbb{R}^{(n-k)\times(n-k)}$, resp.

If A_{11}, A_{22} are nonsingular so is A. Moreover

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix}.$$

Show this!

<u>Outer Product Expansion</u>

Let $\underline{x}, \underline{y} \in \mathbb{R}^n$.

The <u>inner product</u> of $\underline{x}$ and $\underline{y}$ is the usual dot product from Calc II, III:

$$\underline{x} \cdot \underline{y} := \underline{x}^T \underline{y} = (x_1, x_2, \ldots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

The <u>outer product</u> of $\underline{x}$ and $\underline{y}$ is defined to be the $n \times n$ matrix

$$\underline{x}\, \underline{y}^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} (y_1, y_2, \ldots, y_n) = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{pmatrix}.$$

RE. Show that $\underline{y}\, \underline{x}^T = (\underline{x}\, \underline{y}^T)^T$. (It better, ... right!?)

This can also be performed w/ matrices.

Let $A, B \in \mathbb{R}^{m \times n}$. $AB^T \in \mathbb{R}^{m \times m}$.

RE. Write out the details.

Ex. $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}$

$$AB^T = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} (1, 2, 3) + \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} (2, 4, 1)$$

$$= \begin{pmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 1 \\ 8 & 16 & 4 \\ 4 & 8 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 10 & 10 \\ 10 & 20 & 10 \\ 5 & 10 & 5 \end{pmatrix}$$ $\square$

★ And that does it for Chapter 1!