1.5 Elementary Matrices

Consider the $m \times n$ system $A\underline{x} = \underline{b}$.
Let $M \in \mathbb{R}^{m \times m}$ be non-singular.
Then

$$MA\underline{x} = M\underline{b} \qquad (1)$$

is an equivalent $m \times n$ system.

Any sol'n of $A\underline{x} = \underline{b}$ is clearly a sol'n of $MA\underline{x} = M\underline{b}$. (Can you prove it?)

let $\underline{y}$ be a sol'n of $MA\underline{y} = M\underline{b}$.

Is $\underline{y}$ also a sol'n of $A\underline{y} = \underline{b}$?

Yes! Because M is nonsingular, M^{-1} exists and

$$M^{-1}(MA\underline{y}) = M^{-1}(M\underline{b})$$

$$\Rightarrow IA\underline{y} = I\underline{b}$$

$$\Rightarrow A\underline{y} = \underline{b}.$$

Our goal: transform the system $A\underline{x} = \underline{b}$ into one that is simpler to solve by multiplying each side of the eq'n by a sequence of nonsingular matrices $E_1, \ldots, E_k$.

The new system will be of the form $U\underline{x} = \underline{c}$, where

$$U = E_k E_{k-1} \cdots E_2 E_1 A \quad \text{and} \quad \underline{c} = E_k \cdots E_1 \underline{b}$$

Elementary Matrices

An <u>elementary matrix</u> is the result of performing exactly one row operation to the identity matrix I.

There are 3 types:

<u>Type I:</u> Interchange two rows

<u>Ex.</u> $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$E_1 A = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$ $\quad A E_1 = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{pmatrix}$

<u>left</u> multiplication by E_1 results in the <u>row</u> op. of exchanging two rows.

<u>right</u> mult. by E_1 results in the <u>column</u> op. of interchanging two columns.

<u>Type II:</u> Multiply a row by a non zero number.

<u>Ex.</u> $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$E_1 A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 21 & 24 & 27 \end{pmatrix}$ $\quad A E_1 = \begin{pmatrix} 1 & 2 & 9 \\ 4 & 5 & 18 \\ 7 & 8 & 27 \end{pmatrix}$

<u>Type III</u>. Add a multiple of one row to another, and replace one of the rows.

<u>Ex.</u> $E_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$E_3 A = \begin{pmatrix} 22 & 26 & 30 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\quad A E_3 = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 5 & 18 \\ 7 & 8 & 30 \end{pmatrix}$

<u>Thm.</u> If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

<u>Proof.</u> pp. 60-61 in the book. Read it! $\square$

<u>Def'n.</u> A matrix B is <u>row equivalent</u> to a matrix A iff there exists a finite sequence $E_1, \dots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A.$$

In particular:
If two augmented matrices $(A|\underline{b})$ and $(B|\underline{c})$ are row equivalent, then the systems $A\underline{x} = \underline{b}$ and $B\underline{x} = \underline{c}$ are equivalent.

<u>Thm.</u> Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent.

1. A is nonsingular
2. $A\underline{x} = \underline{0}$ has only the trivial sol'n $\underline{x} = \underline{0}$.
3. A is row equivalent to I.

Proof. p. 62. Read it!

Not in the book: 3.) $\Rightarrow$ 2.)

Suppose A is row equiv. to I. Then $(A|\underline{0})$ is row equiv. to $(I|\underline{0})$.
Examining the augmented matrix $(I|\underline{0})$, it is easy to see that $\underline{x}=\underline{0}$ is the only sol'n.

1.) $\Rightarrow$ 3.) :

If A is nonsingular, then $A^{-1}A = I$, so A is row equivalent to I.

Prove 2.) $\Rightarrow$ 1.)?

$\square$

Cor. The system $A\underline{x} = \underline{b}$ of n eq'ns in n unknowns has a unique sol'n if and only if A is nonsingular.
! most important word!

Proof. If A is nonsingular and $\underline{y}$ is any solution of $A\underline{x} = \underline{b}$, then $A\underline{y} = \underline{b}$ and

$$\underline{y} = A^{-1}\underline{b}.$$

Conversely, suppose $A\underline{x} = \underline{b}$ has a unique solution $\underline{y}$. Claim that A is singular. In this case, by the previous thm, the system $A\underline{x} = \underline{0}$ would have a nontrivial sol'n $\underline{z} \neq \underline{0}$. But then $\underline{w} = \underline{y} + \underline{z}$ would be a sol'n to $A\underline{x} = \underline{b}$. Indeed,

$$A\underline{w} = A(\underline{y} + \underline{z}) = A\underline{y} + A\underline{z} = \underline{b} + \underline{0} = \underline{b}$$

but $\underline{y} \neq \underline{w}$.

This is a contradiction! Therefore A must be nonsingular. $\square$

Now, if A is nonsingular, then A is row equiv. to I, and

$$E_k E_{k-1} \cdots E_2 E_1 A = I, \quad \text{and}$$
$$E_k E_{k-1} \cdots E_2 E_1 I = A^{-1}$$

To compute A^{-1}, convert the augmented matrix $(A | I)$ to $(I | B)$. Then $B = A^{-1}$!

Ex. Find A^{-1} for $A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix} \cdots \longrightarrow A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$

Ex. Solve the system $A\underline{x} = \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix}$ w/ A from the previous Ex.

Sol'n: $\underline{x} = A^{-1}b = A^{-1}\begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{pmatrix}$.

<u>Diagonal and Triangular Matrices</u>

A matrix $A \in \mathbb{R}^{n \times n}$ is called <u>upper triangular</u> if $a_{ij} = 0$ for $i > j$ and <u>lower triangular</u> if $a_{ij} = 0$ for $i < j$.

Ex. $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 1 & 4 & 3 \end{pmatrix}$

upper lower

A matrix is called <u>diagonal</u> if $a_{ij}=0$ for $i \neq j$.

Ex. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Triangular Factorization

An $n \times n$ matrix A can be reduced to strict upper triangular form using only elementary row ops of <u>type III</u>. Each row op corresponds to an elem. matrix. Thus, we can decompose any A into a lower triangular matrix and an upper triangular matrix:

$$A = LU.$$

We illustrate this process by example.

Ex. $A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}$

<u>Goal</u>: Make this upper triangular using <u>only</u> type III ops.

Recall, type III look like $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ (always 1's along diagonal).

$R_2 - \left(\frac{1}{2}\right) R_1 \to R_2$ set $\ell_{21} = \frac{1}{2}$

$R_3 - 2 R_1 \to R_3$ set $\ell_{31} = 2$

$\longrightarrow \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{pmatrix}$

$R_3 + 3R_2 \to R_3$

or $R_3 - (-3) R_2 \to R_3$ set $\ell_{32} = -3$

$$\longrightarrow \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix} = U \quad \text{upper triangular.}$$

~~To ~~ ~~convert A to U, we~~

Now set $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}$ lower triangular.

RE. Verify that $LU = A$.

L is called unit lower triangular because it has 1's along its diagonal.

* Why does this work?

To convert A to U, we performed row operations that correspond to the elementary matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad \text{and } E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

So $E_3 E_2 E_1 A = U$.

Thus, $A = (E_3 E_2 E_1)^{-1} U = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{= L} U$

RE. Verify that $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$,

and $E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}$.