1.4. Matrix Algebra

Thm. Let A, B, C be matrices of appropriate dimension, and $\alpha, \beta \in \mathbb{R}$. The following are true.

1. $A + B = B + A$
2. $(A+B) + C = A + (B+C)$
3. $(AB)C = A(BC)$
4. $A(B+C) = AB + AC$
5. $(A+B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
8. $(\alpha + \beta)A = \alpha A + \beta A$
9. $\alpha(A+B) = \alpha A + \alpha B$

You will have to prove one of these on the first exam!

Ex. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Calculate: $A(BC)$ and $(AB)C$ $\quad \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix}$

$A(B+C)$ and $AB + AC$ $\quad \begin{pmatrix} 1 & 7 \\ 5 & 15 \end{pmatrix}$

Ex. $A^k = \underbrace{AA\cdots A}_{k\text{-times}}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$A^3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$

$\vdots$

$A^k = \ ?$

The Identity Matrix

Defn. The $n \times n$ identity matrix is the matrix $I = (\delta_{ij})$ where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

The 3×3 I is thus $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

I is called the identity matrix because

$$BI = B \quad \text{and} \quad IC = C \quad \text{for all } B, C$$

of appropriate dimensions.

The column vectors of I are called the standard basis vectors for $\mathbb{R}^n$.

$$I = (e_1, e_2, e_3, \ldots, e_n)$$

Matrix Inversion

Defn. A $n \times n$ matrix A is said to be nonsingular or invertible if there exists a matrix B such that $BA = AB = I$.

The matrix $B =: A^{-1}$ is said to be the multiplicative inverse of A.

Thm. If A is nonsingular, then its inverse is unique.

Proof. Suppose B and C are both inverses of A. Then,

$$B = BI = B(AC) = (BA)C = IC = C$$

so, $B = C$ and the inverse of A is unique. $\square$

Ex. $A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{pmatrix}$

Show that $AB = BA = I$.

Ex. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ cannot have an inverse

$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ $\quad BA = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix}$ can never $= I$.

Ex. Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

* Take the augmented matrix $(A \mid I)$ to $(I \mid A^{-1})$.

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \xrightarrow{2R_1 - R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -1/2 \end{array}\right)$$

$R_1 - 3R_2 \to R_1$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & -2 & 3/2 \\ 0 & 1 & 1 & -1/2 \end{array}\right) \quad \text{so, } A^{-1} = \begin{pmatrix} -2 & 3/2 \\ 1 & -1/2 \end{pmatrix}$$

check it.

Defn. A matrix ($n \times n$) is said to be <u>singular</u> if it does not have a multiplicative inverse.

Thm. If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular, and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

Show : $AB(B^{-1}A^{-1}) = I$. $\square$

<u>Algebraic rules for transposes</u>

1. $(A^T)^T = A$
2. $(\alpha A)^T = \alpha A^T$
3. $(A+B)^T = A^T + B^T$
4. $(AB)^T = B^T A^T$

Proofs of 1.-3. are straight forward.
For the proof of 4. see p. 53 of the book.

RE 24. A matrix is called <u>idempotent</u> if $A^2 = A$.

Show that $\begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$ is idempotent.