1.3: Matrix Arithmetic

Notation $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ $a_{ij} \in \mathbb{R}$

A is an $m \times n$ matrix.

For the sake of laziness, we may refer to A as

$$A = (a_{ij}).$$

Vectors

For now, a vector is a ordered n-tuple of real numbers. Also, an n-vector.
Written as a $1 \times n$ matrix, we call it a row vector.
Written as an $n \times 1$ matrix, we call it a column vector.

* Column vectors are the default, and will frequent be called simply "vectors".

Def'n. The set of all $n \times 1$ vectors is called Euclidean n-space and is denoted by either $\mathbb{R}^n$ or $\mathbb{E}^n$.

Notation : $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\vec{x} = (x_1 \; x_2 \; \cdots \; x_n)$ or $(x_1, x_2, \ldots, x_n)$

A matrix A can be written as a column vector of its row vectors, or a row vector of its columns.

$$A = (a_{ij}) = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{pmatrix} = (a_1, a_2, \ldots, a_n)$$

Ex. $A = \begin{pmatrix} 3 & 2 & 5 \\ -1 & 8 & 4 \end{pmatrix}$ $a_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$, $a_3 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

$\vec{a}_1 = (3, 2, 5)$ $\vec{a}_2 = (-1, 8, 4)$

Def'n. Two $m \times n$ matrices A and B are said to be <u>equal</u> iff $a_{ij} = b_{ij}$ for all i, j.

<u>Scalar Multiplication</u>

define this.

$\alpha \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$, then (αA) is defined and $(\alpha A) \in \mathbb{R}^{m \times n}$.

Def'n. The entries of αA are (αa_{ij}) for all i, j.

Ex. $A = \begin{pmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{pmatrix}$, $\alpha = \frac{1}{2}$

$\alpha A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 4 & 5 \end{pmatrix}$

<u>Matrix Addition</u>

Def'n. $A, B \in \mathbb{R}^{m \times n}$. The <u>sum</u> $A + B \in \mathbb{R}^{m \times n}$ with entries $(A+B)_{ij} = a_{ij} + b_{ij}$. for all i, j.

i.e., addition is done "component wise".

Ex. $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$

Some Identities

Let 0 be the matrix in $\mathbb{R}^{m\times n}$ w/ all entries 0.
Then,

$$A + 0 = 0 + A = A, \quad \text{and} \qquad \text{additive id.}$$

$$A + (-1)A = (-1)A + A = 0 \qquad \text{additive inverse}$$

We write $(-1)A = -A$.

Now, the "good stuff."

Matrix Multiplication and Linear systems

One eqn, one unknown: $\quad ax = b \qquad a, b \in \mathbb{R}$

m eqn, n unknowns: $\quad Ax = b \qquad A \in \mathbb{R}^{m\times n},\ x \in V^{n},\ b \in \mathbb{R}^m$

where V^n shall represent the space of "n variables": $x_1, x_2, x_3, \ldots, x_n$.

Better: $x \in \mathbb{R}^n$ is an unknown vector.

Ex. One eqn, many unknowns:

$$3x_1 + 2x_2 + 5x_3 = 4$$

$$\underset{1\times 3}{A = \begin{pmatrix} 3 & 2 & 5 \end{pmatrix}} \qquad \underset{3\times 1}{x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \qquad \underset{1\times 1}{b = 4}$$

Define $\quad Ax = \begin{pmatrix} 3 & 2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3x_1 + 2x_2 + 5x_3$

Ex $A = (a_1 \quad a_2 \quad \cdots \quad a_n) \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$Ax = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

e.g., $A = (2 \quad 1 \quad -3 \quad 4) \qquad x = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}$

$$\begin{aligned} Ax &= 2(3) + 1(2) + (-3)(-1) + 4(2) \\ &= 6 + 2 + 3 + 8 = 19 \end{aligned}$$

Thus $x = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}$ solves the system:

$$2x_1 + x_2 - 3x_3 + 4x_4 = 19$$

We want to extend this idea to systems w/ more equations.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 \cdots + a_{2n} x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &= \vdots \\ a_{m1} x_1 + a_{m2} x_2 \cdots + a_{mn} x_n &= b_m \end{aligned}$$

$$A = \underset{m \times n}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \qquad x = \underset{n \times 1}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} \qquad b = \underset{m \times 1}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

Each row of A corresponds to one equation from the system. Define Ax to be the $\underset{m\times 1}{\text{matrix}}$ whose rows <u>are</u> the equations.

Ex. $A = \begin{pmatrix} -3 & 1 \\ 2 & 5 \\ 4 & 2 \end{pmatrix}$ $\quad x = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\quad$ Find b. $\quad b = \begin{pmatrix} -2 \\ 24 \\ 16 \end{pmatrix}$

Another way:

$$Ax = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

$a_i \in \mathbb{R}^m$ and $x_i \in \mathbb{R}$ unknown.

Since $b \in \mathbb{R}^m$, our job is to find coefficients $x_i \in \mathbb{R}$ that make b a <u>linear</u> <u>combination</u> of the columns a_i.

Ex.
$$2x_1 + 3x_2 - 2x_3 = 5$$
$$5x_1 - 4x_2 + 2x_3 = 6$$

$$x_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

verify that $x = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ solves this system.

Thm. A linear system $Ax = b$ is consistent iff b can be written as a linear combination of the column vectors of A.

Ex
$$x_1 + 2x_2 = 1$$
$$2x_1 + 4x_2 \quad 1$$

is inconsistent.

Matrix Multiplication

Def'n. If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the product $AB = C = (c_{ij})$ is the $m \times r$ matrix whose entries are defined by

$$c_{ij} = \vec{a}_i b_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

i.e., the matrix A is multiplied by each column of B separately.

Ex. $A = \begin{pmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{pmatrix}$ $\quad B = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{pmatrix}$

$AB = \begin{pmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{pmatrix}$ $\qquad AB \neq BA$!

$BA = \begin{pmatrix} -1 & -1 \\ 20 & -22 \end{pmatrix}$

Ex. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$AB = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}$, $\quad BA = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

Transpose of a Matrix

Def'n. The transpose of an $m \times n$ matrix A is the $n \times m$ matrix B defined by

$$b_{ji} = a_{ij} \qquad i = 1, \dots, m \quad j = 1, \dots, n.$$

The transpose of A is denoted by A^T.

Ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $\quad C^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

RE.14. Let A be an $m \times n$ matrix. Explain why the multiplications A^TA and AA^T are always possible.

RE.15. A matrix is called skew-symmetric if $A^T = -A$. Show that if A is skew-symmetric, its diagonal entries must be 0.