1.2 : Row Echelon Form

Defn. A matrix is said to be in row echelon form if

1. The first nonzero entry of every row is 1
2. If row k does not consist of entirely zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zeros in row k
3. If there are rows consisting of all zeros, they are below the rows having nonzero entries.

Ex. (REF) : $\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Ex. Not REF : $\begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Defn. The process of using Elementary Row Operations I, II, and III to transform a linear system into one whose augmented matrix is in REF is called Gaussian Elimination.

Ex.
$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) \overset{\cdots}{\longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Overdetermined Systems

A linear system is said to be overdetermined if there are more equations than unknowns. (if $m > n$)
Overdetermined systems are "usually", but not always, inconsistent.

Exs. a)
$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ -x_1 + 2x_2 &= -2 \end{aligned} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right)$$
inconsistent

b)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 5 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
consistent

c)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= 3 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
consistent

Underdetermined Systems

$(m < n)$: more unknowns than equations

If consistent, they must have infinitely many solutions.

"Usually" consistent.

Exs. a) $$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 2x_3 &= 3 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$ inconsistent

b) $$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned} \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$ consistent

Reduced Row Echelon Form (RREF):

AKA Gauss - Jordan Reduction.

Ex. $$\begin{aligned} -x_1 + x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

RREF Rules:

Defn. A matrix is said to be in reduced row echelon form if

1. The matrix is in row echelon form
2. The first nonzero entry in each row is the only nonzero entry in its column.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\cdots \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

Homogeneous Systems

A linear system is said to be homogeneous if the constants on the right hand side are all zero.

RE. Homogeneous systems are always consistent. Prove it.

Thm. An $m \times n$ homogeneous system of linear equations has a nontrivial solution if $n > m$. (i.e., the system is underdetermined.)

Proof. A ~~system~~ homog. system is always consistent.
An $m \times n$ matrix can have at most m nonzero rows, thus at most m lead variables.
Since $m < n$, there must be at least one free variable.
For each ~~value~~ ~~a~~ assignment of values to the free variables, there is a nonzero solution of the system. $\square$

RE(9)
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right)$$

a) Is it possible for this system to be inconsistent?

b) For what values of β will the system have infinitely many solutions?