Math 511: Linear Algebra
Spring 2014: Course Notes

Introduction

Prerequisite: Calculus II
We actually won't need much more than some basic Calc I in this class, but what is needed is a certain level of mathematical maturity that only comes from doing (and understanding) mathematics.

This class is essentially divided into two parts. The first part deals with matrices and matrix computations. This is fairly straight forward, and many of you will find it easy. The second part deals with more abstract ideas, of which our knowledge from the first part of the course will serve as a nice example. I will try to structure the course so that the examples and exercises from the first few sections foreshadow the more abstract concepts that we will be interested in later in the course.

The "Recommended Exercises" are not really "recommended" if you want to do well in this course. What you get out of this course will be directly proportional to what you put in. I "assign" each recommended exercise for a reason. If you really understand all of the REs, then you should do well in this course.

Finally, most of you are not mathematicians, but I am. You may have to work outside of your comfort zone in this course, but it will only help you in the long term. ☺

Section 1.1: Systems of Linear Equations

A linear equation in n unknowns is an equation of the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

where $a_i, b \in \mathbb{R}$ and x_i are variables.

A linear system of m equations in n unknowns is a system of the form:

$$m \text{ equations} \left\{ \begin{array}{l} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \end{array} \right.$$

n "unknowns"

where $a_{ij}, b_i \in \mathbb{R}$.

Our first job this semester is to solve systems of m equations and n unknowns ($m \times n$ systems)

Ex. a) $x_1 + 2x_2 = 5$
$2x_1 + 3x_2 = 8$

2×2

b) $x_1 - x_2 + x_3 = 2$
$2x_1 + x_2 - x_3 = 4$

2×3

c) $x_1 + x_2 = 2$
$x_1 - x_2 = 1$
$x_1 = 4$

3×2

Solve these, if possible.

<u>Def'n.</u> A system of equations is said to be <u>consistent</u> if it has <u>at least one</u> solution. If a system has <u>no</u> solutions, it is called <u>inconsistent</u>.

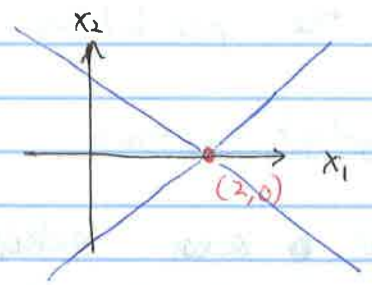
The set of all solutions of a linear system is called the <u>solution set</u>.

2x2 systems

2x2 systems can be thought of as trying to find the point(s) of intersection of two lines in a plane.

<u>Ex.</u> a) $x_1 + x_2 = 2$

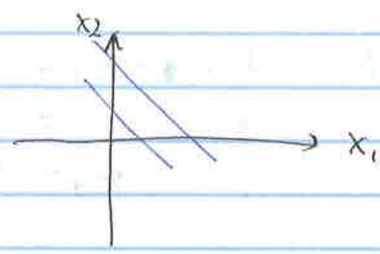
$x_1 - x_2 = 2$



One solution : $(x_1, x_2) = (2, 0)$

b) $x_1 + x_2 = 2$

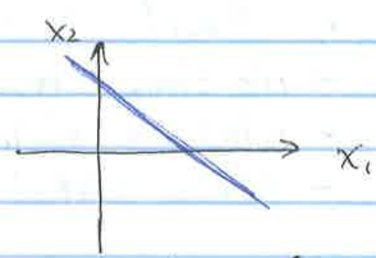
$x_1 + x_2 = 1$



No solutions! Lines are parallel

c) $x_1 + x_2 = 2$

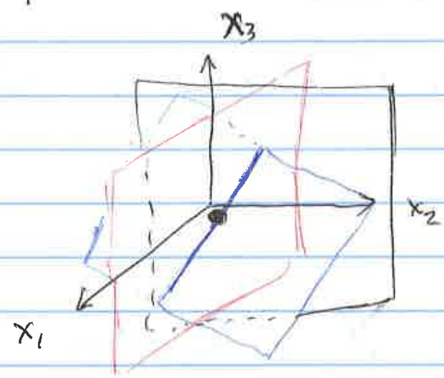
$-x_1 - x_2 = -2$



Infinitely many solutions : $\{(\alpha, 2-\alpha) \mid \alpha \in \mathbb{R}\}$

Equations give the <u>same</u> line.

RE. 11! Give a geometric interpretation of a linear system of 3 equations and 3 unknowns. What kinds of solutions are possible?



All three planes could:
→ intersect in a point
→ intersect in a line
→ not intersect
→ all coincide

So the possibilities are still 0, 1, or ∞ many sol'ns.

Equivalent Systems

Def'n ~~...~~ Two systems of linear equations in n unknowns are said to be equivalent if and only if they have the same sol'n set.

Ex. a)
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 &= 3 \\ 2x_3 &= 4 \end{aligned}$$

b)
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

both have solution $(-2, 3, 2)$.

* We are allowed to:
  - interchange rows
  - multiply a single row by a nonzero constant
  - Add or subtract a multiple of one row to another.

!

Def'n. A system is said to be in strict triangular form if, in the k^{th} equation, the coefficients of the first $k-1$ variables are all 0 and the coefficient $a_k \neq 0$. $k = 1, \dots, n$.

Ex.
$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

This is desirable because it's now easy to solve via "back substitution".

Solve it: $(-3, 4, 2)$

? **Ex.**
$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 2x_4 &= 1 \\ x_2 - 2x_3 + 3x_4 &= 2 \\ 4x_3 + 3x_4 &= 3 \\ 4x_4 &= 4 \end{aligned}$$

$(1, -1, 0, 1)$

Ex.
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

Solve it: $(3, -2, 4)$

To make the algebraic manipulations of the last example easier, we can use a matrix, or a rectangular array of numbers.

We arrange it as follows for a 3×3 system:

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$$

For the last example:

The <u>coefficient matrix</u> is:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{pmatrix}$$

and the <u>augmented matrix</u> is:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right)$$

The system can be solved by performing the same "row operations" that we've already discussed.

<u>Elementary Row Operations:</u>

I. Interchange Two Rows
II. Multiply a row by a nonzero real number
III. Replace a row by its sum with a multiple of another row.

<u>Ex.</u> Do it

<u>Ex.</u> Solve:

$$\begin{aligned} 4[\text{illegible}] - x_2 - x_3 + x_4 &= [\text{illegible}]4 \\ x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3 \end{aligned}$$

Sol'n: $(2, -1, 3, 2)$