Math 511: Linear Algebra
Good Problems 3

Due: Tuesday, 13 May 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: KEY

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I want to see how you got it!

Good Luck!

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1. Find all solutions of the linear system.

$$\begin{cases} x_1 - x_2 + 3x_3 + 2x_4 = 1 \\ -x_1 + x_2 - 2x_3 + x_4 = -2 \\ 2x_1 - 2x_2 + 7x_3 + 7x_4 = 1 \end{cases}$$

Augmented Matrix method:

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 1 \\ -1 & 1 & -2 & 1 & -2 \\ 2 & -2 & 7 & 7 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right)$$

$R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$R_2 - R_3 \to R_3$

$$\to \left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 0 & -7 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$R_1 - 3R_2 \to R_1$

$x_1 = 4 + \beta + 7\alpha$

$x_2 = \beta$

$x_3 = -1 - 3\alpha$

$x_4 = \alpha$

so $\underline{x} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 7 \\ 0 \\ -3 \\ 1 \end{pmatrix}$ $\alpha, \beta \in \mathbb{R}$.

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2. Let

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 7 \\ 1 & 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 5 \\ 4 & 2 & 7 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ -5 & 3 & 5 \end{pmatrix}.$$

Find elementary matrices E and F such that $EA = B$ and $AF = C$.

$A \to B$: swap rows 2 and 3.

So $\boxed{E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$

$A \to C$: Col 1 $-$ 2 Col 2, replace column 1.

So $\boxed{F = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

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3. Find the LU decomposition of

$$A = \begin{pmatrix} -3 & 4 & -5 \\ 6 & 5 & 2 \\ 3 & 2 & 6 \end{pmatrix}.$$

$$\begin{pmatrix} -3 & 4 & -5 \\ 6 & 5 & 2 \\ 3 & 2 & 6 \end{pmatrix} \to \begin{pmatrix} -3 & 4 & -5 \\ 0 & 13 & -8 \\ 0 & 6 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -3 & 4 & -5 \\ 0 & 13 & -8 \\ 0 & 0 & \frac{61}{13} \end{pmatrix} = U$$

$R_2 + 2R_1 \to R_2 \quad : \quad l_{21} = -2$

$R_3 + R_1 \to R_1 \quad : \quad l_{31} = -1$

$R_3 + \left(\frac{-6}{13}\right) R_2 \to R_3 \quad : \quad l_{32} = \frac{+6}{13}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{-6}{13} & 1 \end{pmatrix}$$

So
$$\boxed{A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & \frac{+6}{13} & 1 \end{pmatrix} \begin{pmatrix} -3 & 4 & -5 \\ 0 & 13 & -8 \\ 0 & 0 & 61/13 \end{pmatrix}}$$

<u>Notice</u>: $\det(A) = \det(L)\det(U) = \det(U) = -3(13)\left(\frac{61}{13}\right) = -183.$

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4. Let

$$A = \begin{pmatrix} x & 1 & 1 \\ 1 & x & -1 \\ -1 & -1 & x \end{pmatrix}.$$

Find $p(x) = \det(A)$, then solve $p(x) = 0$.

$$p(x) = \det(A) = \begin{vmatrix} x & 1 & 1 \\ 1 & x & -1 \\ -1 & -1 & x \end{vmatrix} = x \begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ -1 & x \end{vmatrix} + 1 \begin{vmatrix} 1 & x \\ -1 & -1 \end{vmatrix}$$

$$= x(x^2-1) - \cancel{(x-1)} + \cancel{(-1+x)}$$

$$= x(x^2-1).$$

$p(x) = 0$ when $x(x^2-1) = x(x+1)(x-1) = 0$

so $\boxed{x = 0, -1, 1}$

These are the (only) values of x for which A is singular!

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5. Let

$$A = \begin{pmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \end{pmatrix}.$$

Find bases for $N(A)$, Row(A), and Col(A). What is the rank of A?

$$\begin{pmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \end{pmatrix} \xrightarrow{REF} \begin{pmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 3 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$N(A)$: $\left(\begin{array}{ccccc|c} 1 & 3 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$x_1 = -3\gamma - 2\beta - 3\alpha$
$x_2 = \gamma$
$x_3 = -\alpha - \beta$
$x_4 = \beta$
$x_5 = \alpha$

so $N(A) = \text{span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ and null$(A) = 3$.

Row $(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and rk$(A) = 2$.

<u>* Notice:</u> rk(A) + null(A) = n
$2 + 3 = 5$

Col$(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

<u>*</u>: Go back to <u>original</u> A to get the column space!

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6. Let S be the set of all symmetric 2×2 matrices with real entries. Show that S is a subspace of $\mathbb{R}^{n\times n}$. Find a basis for S. What is the dimension of S?

$$S = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \ a, b, d \in \mathbb{R} \right\}$$

Let $A \in S$, $\alpha \in \mathbb{R}$.

Then $\alpha A = \alpha \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha b & \alpha d \end{pmatrix}$ is symmetric ✓

Let $B = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \in S$.

Then $A + B = \begin{pmatrix} a+e & b+f \\ b+f & d+g \end{pmatrix}$ is symmetric ✓

Therefore S is a subspace of $\mathbb{R}^{2\times 2}$.

$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a basis for S.

Indeed $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = aE_1 + bE_2 + dE_3$.

Thus $\dim(S) = 3$.

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7. Let

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

Find the transition matrix from $V = \{\mathbf{v}_1, \mathbf{v}_2\}$ to $U = \{\mathbf{u}_1, \mathbf{u}_2\}$. Use this matrix to compute the U-coordinates of $\mathbf{z} = 2\mathbf{v}_1 + 3\mathbf{v}_2$.

$$\begin{tikzcd} V \arrow[r, "V"] \arrow[rr, bend right, "S = U^{-1}V"'] & E \arrow[r, "U^{-1}"] & U \end{tikzcd}$$

$$V = \begin{pmatrix} 5 & 4 \\ 2 & 9 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

$$U^{-1} = \frac{1}{7-6}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$$

so $\boxed{S = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 2 & 9 \end{pmatrix} = \begin{pmatrix} 31 & 10 \\ -12 & -3 \end{pmatrix}}$

$$[z]_V = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\boxed{[z]_U = S[z]_V = \begin{pmatrix} 31 & 10 \\ -12 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 32 \\ -33 \end{pmatrix}}$$

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8. Let

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator whose matrix representation with respect to the ordered basis $U = \{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

Find the matrix representing L with respect to the ordered basis $V = \{\mathbf{v}_1, \mathbf{v}_2\}$. Use this new matrix to compute $L((-1,0)^T)$.

$$\begin{tikzcd} V \arrow[r, "V"] \arrow[rr, bend right, "S = U^{-1}V"'] & E \arrow[r, "U^{-1}"] & U \end{tikzcd}$$

$S^{-1} = (V^{-1}U)$

$V = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

$U = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$

$U^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$

$S = U^{-1}V = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 12 & 7 \\ -7 & -4 \end{pmatrix}$

$S^{-1} = \frac{1}{-48+49}\begin{pmatrix} -4 & -7 \\ 7 & 12 \end{pmatrix} = \begin{pmatrix} -4 & -7 \\ 7 & 12 \end{pmatrix}$

So $B = S^{-1}AS = \begin{pmatrix} -4 & -7 \\ 7 & 12 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 12 & 7 \\ -7 & -4 \end{pmatrix} = \begin{pmatrix} -4 & -7 \\ 7 & 12 \end{pmatrix}\begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix}$

$\boxed{B = \begin{pmatrix} -222 & -121 \\ 383 & 226 \end{pmatrix}}$

represents L in V Basis. Yuck.

$\begin{pmatrix} -1 \\ 0 \end{pmatrix} =$

$+v_1 \mp 2v_2 = \begin{pmatrix} +1 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ +2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

So $L\begin{pmatrix} -1 \\ 0 \end{pmatrix} = B\begin{bmatrix} -1 \\ 0 \end{bmatrix}_V = \begin{pmatrix} -222 & -121 \\ 383 & 226 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -222 + 242 \\ 383 - 452 \end{pmatrix} = \boxed{\begin{pmatrix} 20 \\ -69 \end{pmatrix}}$

yuck, again.

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9. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}.$$

Find the eigenvalues and corresponding eigenspaces of A. Show that the eigenspaces are linearly independent ~~and orthogonal with respect to the dot product $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{y}$.~~

Consider the linear transformation $L\mathbf{x} = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. Find the matrix representation of L with respect to the basis of $\mathbb{R}^3$ determined by the eigenspaces of A.

$(\lambda-3)(\lambda+1) = \lambda^2 - 2\lambda - 3 - 5$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 5 & -(1+\lambda) \end{vmatrix} = (1-\lambda)\left[-(3-\lambda)(1+\lambda) - 5\right] \qquad \lambda^2 - 2\lambda - 8$$

$= -(1-\lambda)(3-\lambda)(1+\lambda) - 5(1-\lambda)$

$= -(\lambda-1)(\lambda-3)(\lambda+1) - 5 + 5\lambda$

$= -(\lambda^2 - 4\lambda + 3)(\lambda+1) - 5 + 5\lambda$

$= -(\lambda^3 - 3\lambda^2 - \lambda + 3) - 5 + 5\lambda$

$= -\lambda^3 + 3\lambda^2 + 8\lambda - 8$

$= (1-\lambda)(\lambda^2 - 2\lambda - 8)$

So $\lambda_2 = 1$

$\lambda_{1,3} = \dfrac{2 \pm \sqrt{4 - 4(-8)}}{2}$

$= 1 \pm 3 = -2, 4$

Damn, I did that the hard way!.

$\lambda_1 = -2, \quad \lambda_2 = 1, \quad \lambda_3 = 4$

$$N(A - \lambda_1 I): \left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 3 & 0 & 3/2 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} x_1 = -\frac{1}{2}\alpha \\ x_2 = -\frac{1}{5}\alpha \\ x_3 = \alpha \end{array}$$

put $\alpha = -10$, $\boxed{\underline{x}_1 = \begin{pmatrix} 5 \\ 2 \\ -10 \end{pmatrix}}$

$$N(A - \lambda_2 I): \left(\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 5 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 0 & 5 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so $x_1 = \alpha$, $x_2 = 0$, $x_3 = 0$, $\Rightarrow$ $\boxed{\underline{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$$N(A - \lambda_3 I): \left(\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -3 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = -\frac{1}{3}\alpha$, $x_2 = \alpha$, $x_3 = \alpha$ — choose $\alpha = 3$

$\boxed{\underline{x}_3 = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}}$

over

Put $X = (\underline{x}_1, \underline{x}_2, \underline{x}_3) = \begin{pmatrix} 5 & 1 & -1 \\ 2 & 0 & 3 \\ -10 & 0 & 3 \end{pmatrix}$

$\underline{x}_1, \underline{x}_2, \underline{x}_3$ are linearly independent iff $\det(X) \neq 0$.

$\det(X) = -1 \begin{vmatrix} 2 & 3 \\ -10 & 3 \end{vmatrix} = -1(6+30) = -36 \neq 0.$ ✓

Now define $L(\underline{x}) = A\underline{x}$ for every $\underline{x} \in \mathbb{R}^3$.

The matrix B that represents L wrt the "eigenbasis" of $\mathbb{R}^3$ determined by A is

$$B = \left(L(\underline{x}_1), L(\underline{x}_2), L(\underline{x}_3) \right)$$

$L(\underline{x}_1) = A\underline{x}_1 = \lambda_1 \underline{x}_1 = -2\,\underline{x}_1$ by defn. $-2\,\underline{x}_1 = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ in the "eigen basis".

similarly,

$L(\underline{x}_2) = \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $L(\underline{x}_3) = \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

So $\boxed{B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.}$

Notice, this is just the diagonalization of A!

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10. Solve the system of linear differential equations

$$\begin{cases} y_1' = 2y_1 - 6y_3 \\ y_2' = y_1 - 3y_3 \\ y_3' = y_2 - 2y_3 \end{cases}$$

with initial data $y_1(0) = y_2(0) = y_3(0) = 2$.

$Y' = AY$ where $A = \begin{pmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix}$ and $Y(0) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

gen. Sol'n is $Y(t) = c_1 e^{\lambda_1 t} \underline{X}_1 + c_2 e^{\lambda_2 t} \underline{X}_2 + c_3 e^{\lambda_3 t} \underline{X}_3$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & -6 \\ 1 & -\lambda & -3 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} + 0 - 6\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (2-\lambda)\left[(-\lambda)(-2-\lambda) + 3\right] - 6$$

$$= (2-\lambda)\left[\lambda^2 + 2\lambda + 3\right] - 6$$

$$= \cancel{2\lambda^2} + 4\lambda + \cancel{6} - \lambda^3 - \cancel{2\lambda^2} - 3\lambda \cancel{-6}$$

$$= -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1)$$

$$= -\lambda(\lambda+1)(\lambda-1)$$

so $\boxed{\lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = 1}$

$\underline{X}_1$: $N(A)$: $\left(\begin{array}{ccc|c} 2 & 0 & -6 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_3 = \alpha$
$x_2 = 2\alpha$
$x_1 = 3\alpha$

so $\boxed{\underline{X}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}}$

$\underline{X}_2$: $N(A+I)$: $\left(\begin{array}{ccc|c} 3 & 0 & -6 & 0 \\ 1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_1 = 2\alpha$
$x_2 = \alpha$
$x_3 = \alpha$

so $\boxed{\underline{X}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}$

$\underline{X}_3$: $N(A-I)$: $\left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 1 & -1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_1 = 6\alpha$
$x_2 = 3\alpha$
$x_3 = \alpha$

so $\boxed{\underline{X}_3 = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}}$

over

so the gen. sol'n is

$$Y(t) = c_1\begin{pmatrix}3\\2\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}2\\1\\1\end{pmatrix} + c_3 e^{t}\begin{pmatrix}6\\3\\1\end{pmatrix}$$

$$Y(0) = c_1\begin{pmatrix}3\\2\\1\end{pmatrix} + c_2\begin{pmatrix}2\\1\\1\end{pmatrix} + c_3\begin{pmatrix}6\\3\\1\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix}$$

now to solve: $\begin{pmatrix}3&2&6\\2&1&3\\1&1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix}$

$$\left(\begin{array}{ccc|c}3&2&6&2\\2&1&3&2\\1&1&1&2\end{array}\right) \to \left(\begin{array}{ccc|c}1&1&1&2\\0&-1&1&-2\\0&-1&3&-4\end{array}\right) \to \left(\begin{array}{ccc|c}1&1&1&2\\0&1&-1&2\\0&0&-2&2\end{array}\right)$$

$\begin{cases} R_2 - 2R_3 \to R_2 \\ R_3 - 3R_3 \to R_3 \\ R_3 \to R_1 \end{cases}$

$R_2 - R_3 \to R_3$
$R_2 \to -R_2$

$c_1 = 2+1-1 = 2$
$c_2 = 2-1 = +1$
$c_3 = -1$

so,

$$\boxed{Y(t) = 2\begin{pmatrix}3\\2\\1\end{pmatrix} + e^{-t}\begin{pmatrix}2\\1\\1\end{pmatrix} - e^{t}\begin{pmatrix}6\\3\\1\end{pmatrix}}$$

is the particular sol'n of the DE.