Math 511: Linear Algebra
Good Problems 2

Due: Thursday, 24 April 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: KEY

Instructions: Complete all 5 problems. Each problem is worth 20 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I want to see how you got it!

Good Luck!

Name:________________________________

1. Let $L : V \to V$ be a linear operator on a vector space V. Define L^n, $n \geq 1$, recursively by

$$\begin{cases} L^1 = L \\ L^{k+1}(\mathbf{v}) = L(L^k(\mathbf{v})) \quad \forall\, \mathbf{v} \in V. \end{cases}$$

Show that L^n is a linear operator on V for each $n \geq 1$.
[Hint: Use Mathematical Induction.]

If A is the matrix representing L, what is the matrix representing L^n? Justify your answer.

Pf. 1.) $L^1 = L$ is linear by assumption

2.) Assume L^k is linear for some $k \geq 1$.

3.) We must show that L^{k+1} is linear.

$$L^{k+1}(\alpha \underline{v}) = L(L^k(\alpha \underline{v})) = L(\alpha L^k(\underline{v})) = \alpha L(L^k(\underline{v})) = \alpha L^{k+1}(\underline{v}).$$

$$\begin{aligned} L^{k+1}(\underline{u}+\underline{v}) = L(L^k(\underline{u}+\underline{v})) &= L(L^k(\underline{u}) + L^k(\underline{v})) \\ &= L(L^k(\underline{u})) + L(L^k(\underline{v})) \\ &= L^{k+1}(\underline{u}) + L^{k+1}(\underline{v}). \end{aligned}$$

$\square$

If $L(\underline{v}) = A\underline{v}$, then $L^2(\underline{v}) = L(L(\underline{v})) = L(A\underline{v}) = AA\underline{v} = A^2\underline{v}$

Proceeding by induction again, $L^k \underline{v} = A^k \underline{v}$.

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2. *a.)* Let $L : V \to W$ be a linear transformation, and let T be a subspace of W. The *inverse image* (or *preimage*) of T, denoted by $L^{-1}(T)$ is defined by

$$L^{-1}(T) := \{\mathbf{v} \in V \mid L(\mathbf{v}) \in T\}$$

Show that $L^{-1}(T)$ is a subspace of V.

b.) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be given on the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by $L(\mathbf{e}_1) = 2\mathbf{e}_2$, $L(\mathbf{e}_2) = -\mathbf{e}_2$, and $L(\mathbf{e}_3) = \mathbf{e}_1 + 2\mathbf{e}_2$. Find $L^{-1}(T)$, where $T = \text{span}\{(1, -1)^T\}$.

a) Pf. Let $\underline{x}, \underline{y} \in L^{-1}(T)$ and $\alpha \in \mathbb{R}$. Suppose $\underline{x} = L(\underline{v})$, $\underline{y} = L(\underline{w})$.

Consider. $\alpha \underline{x} = \alpha L(\underline{v}) = L(\alpha \underline{v})$

so $\alpha \underline{x} \in L^{-1}(T)$.

Consider $\underline{x} + \underline{y} = L(\underline{v}) + L(\underline{w}) = L(\underline{v} + \underline{w})$

so $\underline{x} + \underline{y} \in L^{-1}(T)$.

Therefore $L^{-1}(T)$ is linear. $\square$

b.) $L = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

Solve $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 2 & -1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$ $\quad x_1 = \frac{1}{2}(\alpha - 3)$, $x_2 = \alpha$, $x_3 = -1$

so $\underline{x} = \alpha \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3/2 \\ 0 \\ -1 \end{pmatrix}$ forms a basis for $L^{-1}(T)$.

$\text{span}\{\underline{x}\} = \text{span}\left\{\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 0 \\ -1 \end{pmatrix}\right\} = L^{-1}(T)$.

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3. Let S be the subspace of $C[a,b]$ spanned by $\sin x$ and $\cos x$. Let $D : S \to S$ be the differentiation operator, $Df = f'$. Find the matrix representing D with respect to this basis for S.

Let $J : S \to S$ be the integration operator, $Jf = \int f\,dx$. Find the matrix representing J in this basis for S.

Show that $DJ = JD = I$.

You just proved a special case of the Fundamental Theorem of Calculus! Word!

$D(\sin x) = \cos x$

$D(\cos x) = -\sin x$ so $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$J(\sin x) = -\cos x$

$J(\cos x) = \sin x$ so $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$JD = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$DJ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

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4. Let $A, B \in \mathbb{R}^{n\times n}$. Show that if A is similar to B, then there exist $n \times n$ matrices S and T, with S nonsingular, such that

$$A = ST \quad \text{and} \quad B = TS.$$

Let A and B be similar matrices, and let $\lambda \in \mathbb{R}$. Show that

1. $A - \lambda I$ and $B - \lambda I$ are similar.
2. $\det(A - \lambda I) = \det(B - \lambda I)$.

a) $A \sim B \Rightarrow A = SBS^{-1}$ w/ S nonsingular.

Let $T = BS^{-1}$

Then $ST = SBS^{-1} = A$

$TS = BS^{-1}S = B$ ✓

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b.) $A \sim B \Rightarrow A = SBS^{-1}$

Consider $S(B-\lambda I)S^{-1} = (SB - S\lambda I)S^{-1}$

$= SBS^{-1} - S\lambda I S^{-1}$

$= SBS^{-1} - \lambda SS^{-1}$

$= A - \lambda I$.

$\det(A-\lambda I) = \det(S(B-\lambda I)S^{-1}) = \det(S)\det(B-\lambda I)\det(S^{-1})$

$= \cancel{\det(S)}\det(B-\lambda I)\frac{1}{\cancel{\det(S)}}$

$= \det(B-\lambda I)$.

5. The *trace* of an $n \times n$ matrix A, denoted by $\mathrm{tr}(A)$, is the sum of its diagonal entries. That is,

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Show that

1. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$;
2. if A is similar to B, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

1\. pf. $(AB)_{ii} = \sum_{m=1}^{n} a_{im}b_{mi}$

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\sum_{m=1}^{n} a_{im}b_{mi}$$

meanwhile,

$$(BA)_{mm} = \sum_{i=1}^{n} b_{mi}a_{im} \quad \text{and}$$

$$\mathrm{tr}(BA) = \sum_{m=1}^{n} (BA)_{mm} = \sum_{m=1}^{n}\sum_{i=1}^{n} b_{mi}a_{im} = \sum_{i=1}^{n}\sum_{m=1}^{n} a_{im}b_{mi}$$

$$= \mathrm{tr}(AB). \quad \square$$

2\. $A \sim B \Rightarrow SA = BS$ for S nonsingular.

Then by 1. $\mathrm{tr}(SA) = \mathrm{tr}(AS)$.

But $\mathrm{tr}(SA) = \mathrm{tr}(BS)$ since $SA = BS$.

Something wrong here I think !!

In sigma-not. this says

$$\sum_{i=1}^{n} (AS)_{ii} = \sum_{i=1}^{n} (BS)_{ii}$$

sorry ^^

This can only hold if

Better:

$A \sim B$ means A, B have the same e'values.

$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$
$= \mathrm{tr}(B)$. $\square$