Math 511: Linear Algebra
Good Problems 1

Due: Thursday, 13 March 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 5 problems. Each problem is worth 20 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I want to see how you got it!

Good Luck!

Name:______________________________

1. Liquid benzene burns in the atmosphere. If a cold object is placed directly over the benzene, water will condense on the object and a deposit of carbon will also form on the object. The chemical equation for this reaction is

$$x_1C_6H_6 + x_2O_2 \longrightarrow x_3C + x_4H_2O.$$

(a.) Write a system of equations that must be satisfied for the equation to be balanced (equal numbers of each atom on each side).

(b.) Solve the system to find (x_1, x_2, x_3, x_4).

a.) The system corresponding to this reaction is:

$$\begin{array}{l} H: \\ C: \\ O: \end{array} \begin{cases} 6x_1 = 2x_4 \\ 6x_1 = x_3 \\ x_2 = x_4 \end{cases}$$

b.) This yields the augmented matrix

$$\left(\begin{array}{cccc|c} 6 & 0 & 0 & -2 & 0 \\ 6 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right)$$

Let $x_4 = \alpha$, then $x_1 = \frac{1}{3}\alpha$
$x_2 = \alpha$
$x_3 = 2\alpha$

The solution of the system is $\boxed{\underline{x} = \alpha \begin{pmatrix} 1/3 \\ 1 \\ 2 \\ 1 \end{pmatrix}}$ for any α.

Put $\alpha = 3$, get: $\boxed{C_6H_6 + 3O_2 \longrightarrow 6C + 3H_2O}$

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2. In coding a message, a blank space was represented by 0, an A by 1, B by 2, C by 3, *etc.* The message was collected in a 4×7 matrix, and transformed by multiplying on the left by the matrix

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

The encoded message is hidden in the matrix:

$$M = \begin{pmatrix} -44 & -27 & -21 & -14 & -34 & -32 & -42 \\ 4 & 23 & 19 & -10 & 10 & 32 & 6 \\ 1 & 10 & 4 & 7 & -12 & 0 & -18 \\ -2 & 13 & 13 & -18 & 3 & 21 & 19 \end{pmatrix}$$

What is the original message?

$M = AB$, so $B = A^{-1}M$ (A is nonsingular)

$$A^{-1} = \begin{pmatrix} -1/2 & -1/2 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ -1/2 & -1/2 & 0 & 0 \\ -1/2 & -1/2 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 19 & 25 & 18 & 1 & 3 & 21 & 19 \\ 5 & 0 & 2 & 1 & 19 & 11 & 5 \\ 20 & 2 & 1 & 12 & 12 & 0 & 18 \\ 21 & 12 & 5 & 19 & 0 & 0 & 0 \end{pmatrix}$$

The message is: SYRACUSE BASKETBALL RULES

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3. Let $\gamma(t) = (x_1(t), x_2(t), x_3(t))^T$ be a particle in space. The *velocity vector field* of the particle is given by

$$\dot{\gamma}(t) = \left(\frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t), \frac{dx_3}{dt}(t)\right)^T.$$

The *norm* of a vector $\mathbf{x}$ is given by $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$. The *unit tangent vector field* along a particle γ is defined to be

$$\mathbf{T}(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}.$$

Let $\gamma(t) = (\sin(t), \cos(t), t)^T$ be a particle in space.

(*a*.) Find the velocity field $\dot{\gamma}$ along γ.

(*b*.) Find the unit tangent field $\mathbf{T}$ along γ.

(*c*.) Show that $\|\mathbf{T}(t)\| = 1$ for all t.

What does this particle look like? How are the velocity field and tangent field related?

a.) $\dot{\gamma}(t) = (\cos(t), -\sin(t), 1)^T$

b.) $\|\dot{\gamma}(t)\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$

$$\vec{T} = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} = \frac{1}{\sqrt{2}}(\cos t, -\sin t, 1)^T$$

c.) $\|\vec{T}\| = \left(\frac{1}{\sqrt{2}}\right)\sqrt{\cos^2 t + \sin^2 t + 1} = \frac{1}{\sqrt{2}}(\sqrt{2}) = 1$ for all t.

The particle is a circular helix  .

The velocity and tangent fields are in the exact same direction, but the velocity field is "longer" at each point by a factor of $\sqrt{2}$.

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4. Determine whether the following are subspaces of P_4. If so, prove it. If not, show or explain why.

(a.) The set of all polynomials in P_4 of even degree.

(b.) The set of all polynomials of degree 3.

(c.) The set of all polynomials $p \in P_4$ such that $p(0) = 0$.

(d.) The set of all polynomials in P_4 having at least one real root.

a.) $p \in S$ looks like $p(x) = a_0 + a_2 x^2$

$\alpha p(x) = \alpha a_0 + \alpha a_2 x^2 \in S$

if $q(x) = b_0 + b_2 x^2$, then

$p+q(x) = p(x) + q(x) = (a_0 + b_0) + (a_2 + b_2) x^2 \in S.$

So yes this is a subspace.

b.) $p \in S$ looks like $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $x_3 \neq 0$.

Thus the 0-vector $0 = 0 + 0x + 0x^2 + 0x^3$ is not in S.

So no this is not a subspace.

c.) $\alpha p(0) = \alpha 0 = 0$ ✓

$\alpha (p+q)(0) = \alpha (p(0) + q(0)) = \alpha (0+0) = \alpha 0 = 0$ ✓

so yes this is a subspace.

d.) Let $p(x) = 2x^2$ and $q(x) = 1 - x^2$. These are both in S, but

$(p+q)(x) = x^2 + 1$ does not have a real root.

Hence this is not a subspace.

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5. Prove the following results.

(*a.*) Any finite set of vectors that contains the zero vector must be linearly dependent.

(*b.*) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two vectors in a vector space V, then $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent if and only if (*both directions*!) one is a scalar multiple of the other.

(*c.*) Any nonempty subset of a linearly independent set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is also linearly independent.

a.) Consider the set $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n, \underline{0}\}$. Then

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_n \underline{v}_n + c_{n+1} \underline{0} = \underline{0}$$

is satisfied by putting $c_{n+1} = 1$ and $c_1 = c_2 = \cdots = c_n = 0$. Thus the vectors are linearly dependent. (since $c_{n+1} \neq 0$.)

b.) Suppose $\underline{x}_1$ and $\underline{x}_2$ are linearly dependent. Then $a\underline{x}_1 + b\underline{x}_2 = \underline{0}$ and one of $a, b \neq 0$, say $a \neq 0$. Then we can divide by a and rearrange to obtain

$$\underline{x}_1 = \frac{-b}{a} \underline{x}_2 .$$

Suppose $\underline{x}_1 = c\underline{x}_2$. Then $\underline{x}_1 - c\underline{x}_2 = \underline{0}$, but the coefficient on $\underline{x}_1$ is $1 \neq 0$. Therefore $\underline{x}_1, \underline{x}_2$ are linearly dependent.

c.) Let $\{\underline{v}_1, \ldots, \underline{v}_k, \underline{v}_{k+1}, \ldots, \underline{v}_n\}$ be linearly independent. Then

$$c_1 \underline{v}_1 + \cdots + c_k \underline{v}_k + c_{k+1} \underline{v}_{k+1} + \cdots + c_n \underline{v}_n = \underline{0}$$

implies $c_1 = \cdots = c_n = 0$.

Now suppose $\{\underline{v}_1, \ldots, \underline{v}_k\}$ are linearly dependent. Then it is possible to find $a_1, \ldots, a_k$ <u>not all 0</u> such that $a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k = \underline{0}$.

But then $(a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_k \underline{v}_k) + c_{k+1} \underline{v}_{k+1} + \cdots + c_n \underline{v}_n = \underline{0}$. This is a contradiction, so $\{\underline{v}_1, \ldots, \underline{v}_k\}$ must be linearly independent. □