Math 511: Linear Algebra
Midterm Exam 2

Thursday, 3 April 2014

Name: Key

Instructions: You must complete all problems. Problem 1 is worth 20 points, and the rest are worth 10 points each.

Show *enough* work, and follow the instructions carefully. Write your name on each page. You may *not* use a calculator, or any other electronic device. You may use one 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

Name:________________________________

1. True or False: Read each statement *carefully*, then write **T** or **F** in the space provided. Each statement is worth 2 points.

F 1.) If $A \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}$, then $\det(\alpha A) = \alpha \det(A)$.

$\det(\alpha A) = \alpha^n \det(A)$

T 2.) If $A, B \in \mathbb{R}^{n\times n}$, then $\det((AB)^T) = \det(A)\det(B)$.

T 3.) A triangular matrix is nonsingular if and only if its diagonal entries are all nonzero.

T 4.) Let $A \in \mathbb{R}^{n\times n}$. If $\mathbf{x} \in \mathbb{R}^n$ is nonzero and $A\mathbf{x} = \mathbf{0}$, then $\det(A) = 0$.

T 5.) If S is a subspace of a vector space V, then S is itself a vector space.

T 6.) If $A \in \mathbb{R}^{m\times n}$, then A and A^T have the same rank.

F 7.) If $A \in \mathbb{R}^{m\times n}$, then A and A^T have the same nullity.

$\text{null}(A) = n - \text{rank}(A)$ but $\text{null}(A^T) = m - \text{rank}(A)$

F 8.) Let $A, B \in \mathbb{R}^{n\times n}$. If $AB = 0$, then $\text{rank}(A) + \text{rank}(B) > n$.

must be $\leq n$

F 9.) If the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ span a vector space V, then they are linearly independent.

there may be "extra" vectors

F 10.) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. If $L(\mathbf{x}_1) = L(\mathbf{x}_2)$, then $\mathbf{x}_1 = \mathbf{x}_2$.

e.g., $L(\underline{x}) = \text{proj}_{\underline{e}_1} \underline{x}$

Name:________________________

2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

Find $\det(A)$.

$A = LU$

$\det(A) = \det(L)\det(U)$

$= 1 \cdot 24$

$= 24$

3. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{0}\}$ be a collection of vectors in a vector space V, where $\mathbf{0}$ is the zero-vector. Prove that these vectors are linearly dependent.

$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_n \underline{v}_n + c_{n+1} \underline{0} = \underline{0}$

put $c_1 = c_2 = \cdots = c_n = 0$ and $c_{n+1} = 1$.

Name:____________________________

4. Show that the functions $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent in $C(\mathbb{R})$.

$$W(f,g) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

$e^{2x} \neq 0$ for any $x \in \mathbb{R}$, so f and g are lin. ind.

In particular, $W(0) = e^0 = 1 \neq 0$.

5. Consider the subset of P_3 defined by

$$S = \{p \in P_3 \text{ such that } p(0) = 0 \ \underline{\text{or}}\ p(1) = 0\}.$$

Is S a vector subspace of P_3? Justify your answer (prove or provide a counter-example).

The axiom C2. fails.

e.g., put $p(x) = x^2$ and $q(x) = x-1$

$p(0) = 0$ and $q(1) = 0$, so $p, q \in S$.

but $(p+q)(x) = x^2 + x - 1$

$(p+q)(0) = 1$
$(p+q)(1) = 1$ $\Rightarrow$ $p+q \notin S$.

In general, take p s.t. $p(0) = 0$ but $p(1) \neq 0$ and
q s.t. $q(0) \neq 0$ but $q(1) = 0$.

Name:________________________

For problems **6** and **7**, consider the following matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 10 \\ 1 & 2 & 4 & 6 \end{pmatrix}.$$

6. Find bases for the row and column spaces of A. Clearly label each basis. What is the rank of A?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 10 \\ 1 & 2 & 4 & 6 \end{pmatrix} \xrightarrow{REF} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$Row(A) = span\{(1,2,3,4), (0,0,1,2)\}$ so $rank(A) = 2$

$Col(A) = span\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix}\right\}$

7. Find a basis for the null space $N(A)$. What is the nullity of A?

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = \alpha$ $\quad x_1 = 2\beta - 2\alpha$

$x_4 = \beta$ $\quad x_3 = -2\beta$

So $\underline{x} = \begin{pmatrix} 2\beta - 2\alpha \\ \alpha \\ -2\beta \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

Thus, $N(A) = span\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right\}$ and $null(A) = 2$.

Name:________________________

For problems **8** and **9**, consider the following vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 7 \end{pmatrix}.$$

8. Let $U = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $V = \{\mathbf{v}_2, \mathbf{v}_3\}$. Notice that U and V form two different bases of $\mathbb{R}^2$. (You don't have to verify this.) Find the transition matrix from V to U.

$S: V \to U$ is given by $S = U^{-1}V$

$U = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ so $U^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

$V = \begin{pmatrix} -1 & 1 \\ 2 & 7 \end{pmatrix}$

$$S = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 9 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

9. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator such that

$$L(\mathbf{v}_1) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{and} \quad L(\mathbf{v}_2) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Find the value $L(\mathbf{v}_3)$.

$[\underline{v}_3]_U = U^{-1}\underline{v}_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ by calculation above.

$L(\underline{v}_3) = \begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 17 \end{pmatrix}.$