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Math 511: Linear Algebra
Midterm Exam 1

Thursday, 20 February 2014

Name: KEY

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work, and follow the instructions carefully. Write your name on each page. You may *not* use a calculator, or any other electronic device. You may use one 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

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1. True or False: Read each statement *carefully*, then write **T** or **F** in the space provided. Each statement is worth 2 points.

F *a.)* If $A \in \mathbb{R}^{n\times n}$ is symmetric, then its diagonal entries must be 0.

This is true for skew-symmetric matrices.

T *b.)* If A is a singular matrix, then A can be factored into a product of elementary matrices.

A is row equiv. to I.

T *c.)* If E is an elementary matrix, then E^{-1} is an elementary matrix of the same type.

Proved in class.

T *d.)* If A is row equivalent to I and $AB = AC$, then B must equal C.

A^{-1} exists, so $A^{-1}AB = A^{-1}AC \Rightarrow B = C$.

F *e.)* If $A, B \in \mathbb{R}^{n\times n}$ are nonsingular matrices, then $A + B$ is also nonsingular.

e.g., $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A+B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

2. Find A^2, A^3, and A^k for the matrix given by

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

$A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} = \begin{pmatrix} 2^3 & 2^3 \\ 2^3 & 2^3 \end{pmatrix}$ $\quad 3 = 2(2) - 1$

$A^3 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} = \begin{pmatrix} 32 & 32 \\ 32 & 32 \end{pmatrix} = \begin{pmatrix} 2^5 & 2^5 \\ 2^5 & 2^5 \end{pmatrix}$ $\quad 5 = 2(3) - 1$

$A^k = \begin{pmatrix} 2^{2k-1} & 2^{2k-1} \\ 2^{2k-1} & 2^{2k-1} \end{pmatrix}$

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3. Find all solutions of the linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 3 \\ -x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 3 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 4 & 2 & 8 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 0 & \frac{22}{5} & \frac{44}{5} \end{array}\right)$$

$R_3 + R_1 \to R_3$

$R_2 - 2R_1 \to R_2$

$R_3 + \frac{4}{5} R_2 \to R_3$

$5R_3 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 0 & 22 & 44 \end{array}\right)$$

Now back solve.

$22x_3 = 44 \Rightarrow x_3 = 2$

$-5x_2 + 6 = 1$

$-5x_2 = -5 \Rightarrow x_2 = 1$

$x_1 + 2 - 2 = 1 \Rightarrow x_1 = 1$

So,

$\boxed{(x_1, x_2, x_3) = (1, 1, 2)}$

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4. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *idempotent* if $A^2 = A$. Show that if A is idempotent, then $I - A$ is also idempotent.

$$(I-A)^2 = (I-A)(I-A)$$
$$= I^2 - A - A + A^2$$
$$= I^2 - A \; (\cancel{-A + A})$$
$$= I - A$$
$\square$

5. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an *involution* if $A^2 = I$. Show that for any fixed angle θ, the matrix

$$B = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

is an involution.

$$B^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad \square$$

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For problems **6** and **7**, let $A \in \mathbb{R}^{3\times 3}$ be given by

$$A = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 5 & 0 \\ 3 & 7 & 1 \end{pmatrix}.$$

6. Find the inverse A^{-1} of A.

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ 3 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ 3 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 3R_1 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -\frac{3}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 1 & -\frac{3}{2} & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_2 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & -\frac{1}{5} & 1 \end{array}\right) \xrightarrow{R_1 - 2R_2 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{2}{5} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & -\frac{1}{5} & 1 \end{array}\right)$$

so, $$\boxed{A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{2}{5} & 0 \\ 0 & \frac{1}{5} & 0 \\ -\frac{3}{2} & -\frac{1}{5} & 1 \end{pmatrix}}$$

7. Solve the system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{x} = A^{-1}\underline{b} = \begin{pmatrix} \frac{1}{2} & -\frac{2}{5} & 0 \\ 0 & \frac{1}{5} & 0 \\ -\frac{3}{2} & -\frac{1}{5} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -1/5 \\ 6/5 \end{pmatrix}$$

so $$\boxed{\underline{x} = \frac{1}{5}\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -1/5 \\ 6/5 \end{pmatrix}}$$

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For problems **8** and **9**, let

$$A = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix},$$

where all four block submatrices are $n \times n$.

8. Find the block form of the inverse A^{-1}.

$$\left(\begin{array}{cc|cc} I & 0 & I & 0 \\ B & I & 0 & I \end{array}\right) \to \left(\begin{array}{cc|cc} I & 0 & I & 0 \\ 0 & I & -B & I \end{array}\right)$$

$R_2 - BR_1 \to R_2$

so $\boxed{A^{-1} = \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix}}$

9. Find the block form of $\frac{1}{2}(A + A^{-1})A^T$.

$$\frac{1}{2}(A + A^{-1})A^T = \frac{1}{2}\left(\begin{pmatrix} I & 0 \\ B & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix}\right)\begin{pmatrix} I & B^T \\ 0 & I \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix}\begin{pmatrix} I & B^T \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & B^T \\ 0 & I \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} I & B^T \\ 0 & I \end{pmatrix}}$$

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10. Find the LU factorization of

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{pmatrix}.$$

Clearly identify the matrices L and U.

$$\begin{pmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{pmatrix}$$

$R_2 + 2R_1 \to R_2 \Rightarrow l_{21} = -2$

$R_3 - 3R_1 \to R_2 \Rightarrow l_{31} = 3$

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{pmatrix}$$

$R_3 + 2R_2 \to R_3 \Rightarrow l_{32} = -2$

$$\boxed{\begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} = U} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} = L}$$

thus $$\boxed{A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}}$$