Name: Key

M511: Linear Algebra (Fall 2017)
Instructor: Justin Ryan
Chapter 3 Exam
Due by: ______________________

WICHITA STATE UNIVERSITY

Read and follow all instructions. You may not use any notes or electronic devices. You may use a compass, straightedge, and colored pens/pencils.

Part I: True/False [4 points each]

Neatly write **T** *on the line if the statement is always true, and* **F** *otherwise [2 points]. In the space provided, give sufficient explanation of your answer [2 points].*

__T__ **1.** Let $\mathbf{x} = (17,-4,2)^T$, $\mathbf{v}_1 = (2,1,-3)^T$, and $\mathbf{v}_2 = (1,-2,4)^T$ in $\mathbb{R}^3$. Then $\mathbf{x}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\left(\begin{array}{cc|c} 2 & 1 & 17 \\ 1 & -2 & -4 \\ -3 & 4 & 2 \end{array}\right) \xrightarrow[R_1 \leftrightarrow R_2,\ R_1 - 2R_2 \to R_1]{3R_2 + R_3 \to R_3} \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 5 & 25 \\ 0 & -2 & -10 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{array}\right) \xrightarrow{R_1 + 2R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right)$$

so $\bar{x} = 6\bar{v}_1 + 5\bar{v}_2$.

__F__ **2.** The vectors $\{(2,3,1)^T, (1,-1,2)^T, (7,3,8)^T\}$ span $\mathbb{R}^3$.

$$\det\begin{pmatrix} 2 & 1 & 7 \\ 3 & -1 & 3 \\ 1 & 2 & 8 \end{pmatrix} = 2\begin{vmatrix} -1 & 3 \\ 2 & 8 \end{vmatrix} - 1\begin{vmatrix} 3 & 3 \\ 1 & 8 \end{vmatrix} + 7\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 2(-14) - 1(21) + 7(7)$$

$$= -28 - 21 + 49 = 0$$

__T__ **3.** Suppose V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form a basis for V. Then $\{\underbrace{\mathbf{v}_1 - \mathbf{v}_2}_{u_1}, \underbrace{\mathbf{v}_2 - \mathbf{v}_3}_{u_2}, \underbrace{\mathbf{v}_3 - \mathbf{v}_4}_{u_3}, \underbrace{\mathbf{v}_4}_{u_4}\}$ also form a basis for V.

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$ $\det(u) = 1 \neq 0$, so u forms a basis.

__F__ **4.** Any collection of four polynomials in P_4 form a spanning set for P_4.

The polynomials must be linearly independent.

__F__ **5.** $S = \text{span}\{x, x^2, x|x|\}$ is a subspace of $C(\mathbb{R})$ of dimension 3.

x^2 and $x|x|$ are linearly dependent: $W(x^2, x|x|) = \det\begin{pmatrix} x^2 & x|x| \\ 2x & 2|x| \end{pmatrix} = 2x^2|x| - 2x^2|x| = 0$.

Part I: Written Problems [10 points each]
Follow all instructions exactly, and show enough work.

6 – 9. For questions 6 through 9, consider the vector space P_4 with standard (ordered) basis $E = \{1, x, x^2, x^3\}$, and $V = \{1, (x-2), (x-2)^2, (x-2)^3\}$.

6. Show that V forms a basis of P_4.

$$V = \begin{pmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\det(V) = 1 \neq 0$, so the elements of V are lin. ind.

Since there are 4, they must also span P_4.

7. Find the transition matrix from the ordered basis E to V.

Recall $V = \begin{pmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$: $V \to E$

Then $V^{-1} = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$: $E \to V$

8. Write the vector $p(x) = 8 - 12x + 6x^2 - x^3$ in V-coordinates.

$$[p]_V = V^{-1}[p]_E$$

$$= \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ -12 \\ 6 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 - 24 + 24 - 8 \\ 0 - 12 + 24 - 12 \\ 0 + 0 + 6 - 6 \\ 0 + 0 + 0 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

So $p(x) = -1(x-2)^3$.

9. Find an ordered basis $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ of P_4 for which

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the transition matrix from V to U. Give your answer as polynomials in P_4 (**not** in coordinates).

$S = U^{-1}V : \; V \to U$

so $U = VS^{-1}$

$$= \begin{pmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 9 & -27 \\ 0 & 1 & -6 & 27 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so,

$$\begin{cases} u_1(x) = 1 \\ u_2(x) = x-3 \\ u_3(x) = x^2-6x+9 = (x-3)^2 \\ u_4(x) = x^3-9x^2+27x-27 = (x-3)^3 \end{cases}$$

Compute S^{-1}:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

10. Consider the following vectors in $\mathbb{R}^2$,

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Write $\mathbf{x} = 4\mathbf{u}_1 - 2\mathbf{u}_2$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$[x]_u = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ $\quad$ $[x]_v = S[x]_u$ where $S: u \to v$ is the transition matrix.

$S = V^{-1}U$.

$U = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ $\quad$ $V = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$, $\quad V^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

$$S = V^{-1}U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 11 \\ 4 & 4 \end{pmatrix}$$

$$[x]_v = \begin{pmatrix} 10 & 11 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \end{pmatrix}_v$$

So $\boxed{\bar{x} = 18\bar{v}_1 + 8\bar{v}_2}$

11. Consider the matrix,

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 6 & -3 & -5 & -7 \end{pmatrix}$$

Find bases for the row space, column space, and null space of A. Clearly label each answer.

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 6 & -3 & -5 & -7 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ 6 & -3 & -5 & -7 \end{pmatrix} \xrightarrow[\substack{R_3 - 6R_1 \to R_3 \\ R_2 + 3R_1 \to R_2}]{} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & -15 & 1 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 7 & 5 \end{pmatrix}$$

So, $\text{row}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 5 \end{pmatrix} \right\}$

and. $\text{col}(A) = \mathbb{R}^3$

Null space:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 6 & 4 & 0 \\ 0 & 0 & 7 & 5 & 0 \end{array}\right)$$

$x_4 = t$

$x_3 = -\frac{5}{7} t$

$x_2 = -6x_3 - 4t = +\frac{30-28}{7} t = \frac{2}{7} t$

$x_1 = -x_3 = \frac{5}{7} t$

So $\text{Null}(A) = \text{span}\left\{ \begin{pmatrix} 5 \\ 2 \\ -5 \\ 7 \end{pmatrix} \right\}$

12. Consider a matrix $A \in \mathbb{R}^{3\times 5}$ whose columns $\mathbf{a}_1, \ldots, \mathbf{a}_5$ satisfy $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_5$ are linearly independent, and $\mathbf{a}_3 = \mathbf{a}_1 - \mathbf{a}_2$ and $\mathbf{a}_4 = 2\mathbf{a}_1 + \mathbf{a}_3$.

(*a*) What is the reduced row echelon form (RREF) of A?

(*b*) What is the column space of A?

a) In a-coords, $\bar{a}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\bar{a}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $\bar{a}_5 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Thus, $\bar{a}_3 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $\bar{a}_4 = \begin{pmatrix}3\\-1\\0\end{pmatrix}$.

The rref form of A is thus $\begin{pmatrix}1 & 0 & 1 & 3 & 0\\ 0 & 1 & -1 & -1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$

b) $\text{col}(A) = \text{span}\{\bar{a}_1, \bar{a}_2, \bar{a}_5\} = \mathbb{R}^3$.

13. Let $b \in \mathbb{R}$ be any real number, and define the operations $\oplus_b$ and $\otimes_b$ on $\mathbb{R}$ by:

$$\begin{aligned} x \oplus_b y &= x + y - b, \quad \text{and} \\ \alpha \otimes_b x &= \alpha x + (1-\alpha) b \end{aligned}$$

Is $(\mathbb{R}, \oplus, \otimes)$ a vector space? Give sufficient proof of your answer.

Yes. All 10 Axioms check out.