Name: Key

M511: Linear Algebra (Fall 2017)
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Chapter 5 Exam

WICHITA STATE UNIVERSITY

Read and follow all instructions. You may not use any notes or electronic devices. Show enough work.

1. **[20 points]** Consider the subspace S of $\mathbb{R}^3$ spanned by $\mathbf{x} = (1, -1, 1)^T$. Find an orthonormal basis for $S^{\perp}$.

$\bar{x} = (x, y, z)^T \in S^{\perp}$ must satisfy $x - y + z = 0$. Put $z = t$, $y = u$ then $x = u - t$

So, $S^{\perp} = \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\bar{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\|\bar{v}_1\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

So, $\bar{u}_1 = (-1/\sqrt{2}, 0, 1/\sqrt{2})^T$

$\bar{p}_1 = \text{Proj}_{\bar{u}_1} \bar{v}_2 = -1/\sqrt{2} \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$

$\bar{r}_2 = \bar{v}_2 - \bar{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}$

$\|\bar{r}_2\| = \sqrt{1/4 + 4/4 + 1/4} = \sqrt{3/2}$

$\bar{u}_2 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

$$\boxed{S^{\perp} = \text{span}\left\{ \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \right\}}$$

2. Let $(V, \langle\ ,\ \rangle)$ be an inner product space.

(a) [10 points] Prove the Cauchy-Schwarz-Bunyakovsky Inequality: $|\langle \mathbf{x}, \mathbf{y}\rangle| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$.

Case I. If $\bar{y}=\bar{0}$, the (in)equality is trivial.

Case II. If $\bar{y}\neq\bar{0}$, consider $\bar{p}=\text{proj}_{\bar{y}}\,\bar{x}$.

The Pythagorean Law applies to $\bar{p}$ and $\bar{x}-\bar{p}$.

Thus,

$$\|\bar{x}\|^2 = \|\bar{x}-\bar{p}\|^2 + \|\bar{p}\|^2$$

or,

$$\|\bar{p}\|^2 = \|\bar{x}\|^2 - \|\bar{x}-\bar{p}\|^2$$

$$\|$$

$$\left(\frac{\langle \bar{x},\bar{y}\rangle}{\|\bar{y}\|}\right)^2 = \|\bar{x}\|^2 - \|\bar{x}-\bar{p}\|^2$$

$$\Rightarrow \quad (\langle \bar{x},\bar{y}\rangle)^2 = \|\bar{x}\|^2\|\bar{y}\|^2 - \|\bar{x}-\bar{p}\|^2\|\bar{y}\|^2$$

$$\le \|\bar{x}\|^2\|\bar{y}\|^2$$

Taking the square root of both sides yields, $|\langle \bar{x},\bar{y}\rangle| \le \|\bar{x}\|\,\|\bar{y}\|$ ∎

(b) [10 points] Use part **(a)** to prove the triangle inequality: $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

$$\|\bar{x}+\bar{y}\|^2 = \langle \bar{x}+\bar{y}, \bar{x}+\bar{y}\rangle$$

$$= \langle \bar{x},\bar{x}\rangle + 2\langle \bar{x},\bar{y}\rangle + \langle \bar{y},\bar{y}\rangle$$

$$\le \langle \bar{x},\bar{x}\rangle + 2|\langle \bar{x},\bar{y}\rangle| + \langle \bar{y},\bar{y}\rangle$$

$$\le \|\bar{x}\|^2 + 2\|\bar{x}\|\,\|\bar{y}\| + \|\bar{y}\|^2 \quad \text{by CSB}$$

$$= (\|\bar{x}\| + \|\bar{y}\|)^2$$

Taking the square root of both sides yields

$$\|\bar{x}+\bar{y}\| \le \|\bar{x}\| + \|\bar{y}\|.$$

3. **[20 points]** Consider the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 2 \end{pmatrix}.$$

Use the Gram-Schmidt process to find an orthonormal basis for col(A).

$\bar{v}_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ $\qquad \bar{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

$\|\bar{v}_1\| = \sqrt{3^2+4^2+0^2} = \sqrt{25} = 5$

$\bar{u}_1 = (3/5, 4/5, 0)^T$

$\bar{p}_1 = \text{proj}_{\bar{u}_1} \bar{v}_2 = \left(\frac{-3}{5} + \frac{8}{5}\right) \bar{u}_1 = \bar{u}_1 = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$

$\bar{r}_2 = \bar{v}_2 - \bar{p}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} = \begin{pmatrix} -8/5 \\ 6/5 \\ 10/5 \end{pmatrix}$

$\|\bar{r}_2\| = \frac{1}{5}\sqrt{8^2+6^2+10^2} = \frac{1}{5}\sqrt{200} = 2\sqrt{2}$

$\bar{u}_2 = \frac{1}{10\sqrt{2}} \begin{pmatrix} -8 \\ 6 \\ 10 \end{pmatrix} = \frac{1}{5\sqrt{2}} \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -4/5 \\ 3/5 \\ 1 \end{pmatrix}$

thus, $\boxed{\text{col}(A) = \text{span}\left\{ \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -4/5 \\ 3/5 \\ 1 \end{pmatrix} \right\}}$

4. **[20 points]** Consider the subspace $S = \operatorname{span}\{1, x, x^2\}$ of $C[0,1]$ with inner product defined by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Find an orthonormal basis for S.

$$\|1\|^2 = \int_0^1 1^2\,dx = x\Big|_0^1 = 1$$

so $\boxed{u_1(x) = 1}$

$$\tilde{p}_1 = 1\cdot\int_0^1 x\,dx = 1\cdot\frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$$

$$\tilde{r}_2 = x - 1/2$$

$$\|\tilde{r}_2\|^2 = \int_0^1 (x-1/2)^2\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} u^2\,du = \frac{1}{3}u^3\Big|_{-1/2}^{1/2} = \frac{1}{3}\cdot 2\cdot\frac{1}{8} = \frac{1}{12} \quad\Rightarrow\quad \|\tilde{r}_2\| = \frac{1}{2\sqrt{3}}$$

$$\boxed{u_2(x) = 2\sqrt{3}\,x - \sqrt{3}}$$

$$\tilde{p}_2 = 1\cdot\int_0^1 x^2\,dx + (2\sqrt{3}x - \sqrt{3})\int_0^1 2\sqrt{3}x^3 - \sqrt{3}x^2\,dx$$

$$= \frac{1}{3} + (2\sqrt{3}x - \sqrt{3})\left[\frac{\sqrt{3}}{2}x^4 - \frac{1}{\sqrt{3}}x^3\right]_0^1 = \frac{1}{3} + (2\sqrt{3}\,x - \sqrt{3})\Big(\overbrace{\frac{3}{2\sqrt{3}} - \frac{2}{2\sqrt{3}}}^{1/2\sqrt{3}}\Big) = \frac{1}{3} + x - \frac{1}{2} = x - \frac{1}{6}$$

$$\tilde{r}_3 = x^2 - x + \frac{1}{6} = \left(x^2 - x + 1/4\right) - 1/4 + 1/6 = (x - 1/2)^2 - 1/12$$

$$\|\tilde{r}_3\|^2 = \int_0^1 \left((x-1/2)^2 - 1/12\right)^2 dx = \int_{-1/2}^{1/2} (u^2 - 1/12)^2\,du = 2\int_0^{1/2} (u^2 - 1/12)^2\,du = 2\int_0^{1/2} u^4 - 1/6\,u^2 + 1/144\,du$$

$$= \cancel{2}\cdot\left(\frac{1}{5}\cdot\frac{1}{2^4} - \frac{1}{6}\cdot\frac{1}{3}\cdot\frac{1}{2^2} + \frac{1}{144}\cdot\frac{1}{\cancel{2}}\right) = \frac{1}{80} - \frac{1}{72} + \frac{1}{144} = \frac{1}{80} - \frac{1}{144} = \frac{1}{16\cdot 5} - \frac{1}{16\cdot 9} = \frac{9-5}{16\cdot 9\cdot 5} = \frac{\cancel{4}\,1}{\cancel{16}\,4\cdot 9\cdot 5}$$

$$= \frac{1}{180}$$

$$\|\tilde{r}_3\| = \frac{1}{6\sqrt{5}}$$

so, $u_3(x) = 6\sqrt{5}\left(x^2 - x + 1/6\right) = \boxed{6\sqrt{5}\,x^2 - 6\sqrt{5}\,x + \sqrt{5} = u_3(x)}$

5. Consider the following vectors in $(\mathbb{R}^4, \cdot)$.

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

(a) [10 points] State completely the Pythagorean Law for Inner Product Spaces.

If $\langle \bar{x}, \bar{y} \rangle = 0$, then $\|\bar{x} - \bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2$.

(b) [10 points] Compute $\mathbf{p} = \mathrm{proj}_{\mathbf{y}}\mathbf{x}$, and show that $(\mathbf{x} - \mathbf{p}) \perp \mathbf{p}$.

$$\bar{p} = \mathrm{proj}_{\bar{y}}\bar{x} = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{y}\|^2}\bar{y} = \frac{1(-2)+1(1)+2(2)+2(0)}{(-2)^2+1^2+2^2+0^2}(-2,1,2,0)^T = \frac{3}{9}(-2,1,2,0)^T = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 0\right)^T$$

$$\bar{x} - \bar{p} = (1,1,2,2)^T - \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 0\right)^T = \left(\frac{5}{3}, \frac{2}{3}, \frac{4}{3}, \frac{6}{3}\right)^T$$

$$\langle \bar{p}, \bar{x} - \bar{p} \rangle = -\frac{2}{3}\left(\frac{5}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(\frac{4}{3}\right) + 0\left(\frac{6}{3}\right) = \frac{-10+2+8}{9} = \frac{0}{9} = 0.$$

(c) [10 points] Verify that the Pythagorean Law holds for $\mathbf{x}, \mathbf{p}$, and $\mathbf{x} - \mathbf{p}$.

$$\|\bar{x}\|^2 = 1^2 + 1^2 + 2^2 + 2^2 = 1+1+4+4 = 10$$

$$\|\bar{p}\|^2 = \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + (0)^2 = \frac{4+1+4}{9} = \frac{9}{9} = 1$$

$$\|\bar{x} - \bar{p}\|^2 = \left(\frac{5}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{6}{3}\right)^2 = \frac{25+4+16+36}{9} = \frac{81}{9} = 9$$

Indeed, $10 = 1 + 9$ ✓