Name: Key

M511: Linear Algebra (Fall 2017)
Instructor: Justin Ryan
Chapter 4 Exam

WICHITA STATE UNIVERSITY

Read and follow all instructions. You may not use any notes or electronic devices.

Part I: True/False [4 points each]
Neatly write **T** *on the line if the statement is always true, and* **F** *otherwise [2 points]. In the space provided, give sufficient explanation of your answer [2 points].*

T **1.** If $L : V \to V$ is a linear transformation and $\mathbf{x} \in \ker(L)$, then $L(\mathbf{v}+\mathbf{x}) = L(\mathbf{v})$ for all $\mathbf{v} \in V$.

$L(\bar{v}+\bar{x}) = L(\bar{v}) + L(\bar{x}) = L(\bar{v}) + \bar{0} = L(\bar{v})$.

F **2.** Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. If $L(\mathbf{x}_1) = L(\mathbf{x}_2)$, then $\mathbf{x}_1$ must be equal to $\mathbf{x}_2$.

Only true if L is one-to-one.

F **3.** Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix representation of L. Then $\text{range}(L) = \text{row}(A)$.

$\text{range}(L) = \text{col}(A)$.

T **4.** Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix representation of L. Then $\ker(L) = \text{Null}(A)$.

$L(\bar{x}) = \bar{0} \Rightarrow L(c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n) = \bar{0} \Rightarrow c_1L(\bar{v}_1) + c_2L(\bar{v}_2) + \cdots + c_nL(\bar{v}_n) = \bar{0}$

$\Rightarrow \left(L(\bar{v}_1), L(\bar{v}_2), \ldots, L(\bar{v}_n)\right)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \bar{0} \Rightarrow A\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \bar{0}$.

F **5.** The transformation of $\mathbb{R}^2$ that reflects each point in the plane over the line $y = 2x - 4$ is a linear transformation.

$L(\bar{0}) \neq \bar{0}$!

Part I: Written Problems [10 points each]
Follow all instructions exactly, and show enough work.

6–7. Consider the ordered bases $U = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right\}$ and $V = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \end{pmatrix} \right\}$ of $\mathbb{R}^2$.

6. Consider the linear transformation defined by $L(\mathbf{u}_1) = 2\mathbf{u}_1 - \mathbf{u}_2$ and $L(\mathbf{u}_2) = \mathbf{u}_1 + 3\mathbf{u}_2$. Find the matrix representing L with respect to the basis U.

$$\boxed{A_L = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}}$$

7. Find the matrix representing L with respect to the basis V.

$$B_L = S^{-1} A_L S$$

where $S : V \to U = U^{-1} V$.

$$V = \begin{pmatrix} 1 & -2 \\ 1 & -4 \end{pmatrix}, \quad U = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}, \quad U^{-1} = \frac{1}{4}\begin{pmatrix} -3 & -2 \\ -1 & -2 \end{pmatrix}$$

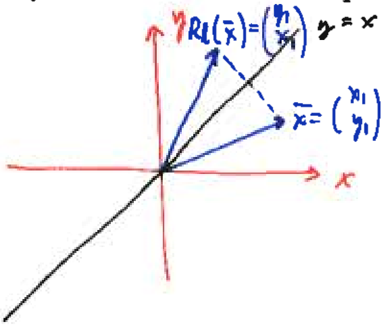
$$S = \frac{1}{4}\begin{pmatrix} -3 & -2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -5 & -2 \\ -3 & -6 \end{pmatrix}$$

$$S^{-1} = \frac{16}{\cancel{\;}} \cdot \frac{1}{24}\begin{pmatrix} -6 & 2 \\ 3 & -5 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} -6 & 2 \\ 3 & -5 \end{pmatrix}$$

$$B = \frac{2}{3}\begin{pmatrix} -6 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\left(\frac{1}{4}\right)\begin{pmatrix} -3 & -2 \\ -1 & -2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -6 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} -7 & -6 \\ 0 & -4 \end{pmatrix}$$

$$\boxed{B_L = \frac{1}{6}\begin{pmatrix} 42 & 28 \\ -21 & 2 \end{pmatrix}}$$

8. Let $R_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects each vector over the line $\ell : y = x$. Find the matrix representing R_ℓ with respect to the standard basis.



$$R_\ell\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$R_\ell\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\boxed{\text{So, } R_\ell = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}}$$

9. Let $U = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$, and suppose $L(\mathbf{u}_1) = 2\mathbf{u}_1$ and $L(\mathbf{u}_2) = -4\mathbf{u}_2$. Find the matrix representing L with respect to the standard basis.

$$U = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}, \quad U^{-1} = \frac{1}{2}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}$$

$$A_E = \begin{pmatrix}2 & 0\\0 & -4\end{pmatrix}$$

$$B = A_U = U^{-1} A_E U = \frac{1}{2}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & -4\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}2 & -2\\-4 & -4\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}-2 & -6\\-6 & -2\end{pmatrix}$$

$$\boxed{B = \begin{pmatrix}-1 & -3\\-3 & -1\end{pmatrix}}$$

10–11. Suppose $L:\mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}.$$

10. Find a basis for $\ker(L)$.

$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$x_1 = -x_2$

$x_3 = -x_2$

So x_2 = free.

Therefore $\boxed{\ker(L) = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\right\}.}$

11. Is L onto? Explain.

$n = 3$ and $\operatorname{null}(A) = 1$, so $\operatorname{rk}(A) = 2$.

Therefore $\dim(\operatorname{range}(L)) = 2$, and $\operatorname{range}(L) = \mathbb{R}^2$.

12–13. Consider the subspace $S = \operatorname{span}\{e^{-x}\cos x, e^{-x}\sin x\}$ of $C(\mathbb{R})$.

12. Find the matrix representing the derivative $D : f \mapsto f'$ on S.

$D(e^{-x}\cos x) = -e^{-x}\cos x - e^{-x}\sin x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$D(e^{-x}\sin x) = +e^{-x}\cos x - e^{-x}\sin x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So, $D = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$

13. Use the Fundamental Theorem of Calculus to compute

$$\int 3e^{-x}\cos x - 5e^{-x}\sin x\,dx$$

as a matrix product.

$A = D^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

So $\int 3e^{-x}\cos x - 5e^{-x}\sin x\,dx = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$= e^{-x}\cos x + 4e^{-x}\sin x$