Name: Key
M511: Linear Algebra (Fall 2017)
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Chapter 1 Exam

WICHITA STATE UNIVERSITY

Read and follow all instructions. You may not use any notes or electronic devices. You may use a compass, straightedge, and colored pens/pencils.

Part I: True/False [4 points each]
Neatly write **T** *on the line if the statement is always true, and* **F** *otherwise [2 points]. In the space provided, give sufficient explanation of your answer [2 points].*

F **1.** If A and B are $n \times n$ matrices, then $(A - B)^2 = A^2 - 2AB + B^2$.

$AB \neq BA$ in general, so
$(A-B)^2 = A^2 - AB - BA + B^2$

F **2.** If $A = A^{-1}$, then either $A = I$ or $A = I^{-1}$.

Counter example: let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
then $A^2 = I$.

T **3.** Every homogeneous linear system is consistent.

Homogeneous means $A\bar{x} = \bar{0}$, so $\bar{x} = \bar{0}$ is always a solution.

T **4.** If A and B are nonsingular $n \times n$ matrices, then $C = AB$ is also nonsingular.

$C^{-1} = (AB)^{-1} = B^{-1}A^{-1}$.

F **5.** If A and B are nonsingular $n \times n$ matrices, then $C = (A + B)$ is also nonsingular.

Counter example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

This C is singular.

Part I: Written Problems [10 points each]
Follow all instructions exactly, and show enough work.

6. Solve the linear system, if possible, using your favorite method.

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 - x_2 + x_3 = 3 \\ -x_1 + 2x_2 + 3x_3 = 7 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 3 & 7 \end{array}\right)$$

$\left.\begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - (-1)R_1 \to R_3 \end{array}\right\}$;

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 4 & 2 & 8 \end{array}\right)$$

$-\frac{1}{5} R_2 \to R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3/5 & -1/5 \\ 0 & 4 & 2 & 8 \end{array}\right)$$

$R_3 - 4R_2 \to R_3$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3/5 & -1/5 \\ 0 & 0 & 22/5 & 44/5 \end{array}\right)$$

$\frac{5}{22} R_3 \to R_3$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3/5 & -1/5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Now back solve:

$x_3 = 2$

$x_2 = -\frac{1}{5} + \frac{3}{5}(2) = \frac{5}{5} = 1$

$x_1 = 1 - 2(1) + 1(2) = 1$

Solution set:

$\{(1,1,2)\}$.

7. The augmented matrix is in reduced row echelon form. Find the solution of the corresponding system.

$$\left(\begin{array}{cccc|c} 1 & 5 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

x_1, x_4 are lead variables
x_2, x_3 are free.

$x_2 = t$
$x_3 = s$
$x_4 = 6$
$x_1 = 3 - 5x_2 + 2x_3$
$= 3 - 5t + 2s$

Solution Set:

$\{(3-5t+2s, t, s, 6) \mid s, t \in \mathbb{R}\}$

8. Consider the matrices

$$A = \begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix}.$$

Compute AB.

$$AB = \begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 19 \\ 4 & 0 \end{pmatrix}.$$

9. Given that $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, show that $R^{-1} = R^T$.

$$R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$R^T R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $R^T = R^{-1}$. $\blacksquare$

10. The matrix $A = \begin{pmatrix} -2 & 4 \\ 6 & 8 \end{pmatrix}$ is nonsingular. Write A^{-1} as a product of elementary matrices.

$\begin{pmatrix} -2 & 4 \\ 6 & 8 \end{pmatrix}$

$R_2 + 3R_1 \to R_2$

$\begin{pmatrix} -2 & 4 \\ 0 & 20 \end{pmatrix}$

$\frac{1}{20} R_2 \to R_2$

$\begin{pmatrix} -2 & 4 \\ 0 & 1 \end{pmatrix}$

$R_1 - 4R_2 \to R_1$

$\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

$-\frac{1}{2} R_1 \to R_1$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So, $A^{-1} = \underbrace{\begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix}}_{E_4} \underbrace{\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}}_{E_3} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1/20 \end{pmatrix}}_{E_2} \underbrace{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}}_{E_1}$

11. Find the inverse of the given matrix. Do *not* use determinants.

$$A = \begin{pmatrix} 2 & 5 & 1 \\ 1 & 3 & 0 \\ 2 & 5 & 0 \end{pmatrix}$$

$\left(\begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right)$

$R_1 - R_3 \to R_3$

$R_1 - 2R_2 \to R_2$

$\left(\begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$

$R_1 - R_3 \to R_1$

$R_2 - R_3 \to R_2$

$\left(\begin{array}{ccc|ccc} 2 & 5 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$

$R_1 + 5R_2 \to R_1$

$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 5 & -10 & 6 \\ 0 & -1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$

$\frac{1}{2} R_1 \to R_1$

$-R_2 \to R_2$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/2 & -5 & 3 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$

So $A^{-1} = \begin{pmatrix} 5/2 & -5 & 3 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}$

12. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Compute

$$\begin{pmatrix} A^{-1} \\ I \end{pmatrix} \begin{pmatrix} A & I \end{pmatrix}.$$

$$\begin{pmatrix} A^{-1} \\ I \end{pmatrix} \begin{pmatrix} A & I \end{pmatrix} = \begin{pmatrix} A^{-1}A & A^{-1} \\ A & I \end{pmatrix} = \begin{pmatrix} I & A^{-1} \\ A & I \end{pmatrix}.$$

13. Find the *LU* Factorization of the matrix.

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{pmatrix}$$

$$\left.\begin{array}{l} R_2 - (-2)R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}\right\} : \begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{pmatrix}$$

$$R_3 - (-2)R_2 \to R_3 : \begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}$$

So, $$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}}_{U}.$$