Calculus III: Project 7

Due date: Monday, 22 July 2013

Recall from Calculus II (section 7.5) a region $\mathcal{R}$ in the plane can be thought of as a thin plate or lamina. Suppose that $\mathcal{R}$ has constant density ρ; then its *moments* with respect to the y- and x-axes are given by the integrals

$$M_y = \rho \int_a^b x f(x)\, dx, \quad \text{and} \quad M_x = \rho \int_a^b \tfrac{1}{2}[f(x)]^2\, dx, \tag{1}$$

where $\mathcal{R}$ is the region bounded by $x = a, x = b, y = 0,$ and $y = f(x)$.

Now, in section 12.4, we want to consider a more general region $\mathcal{R}$ with a varying density $\rho = \rho(x, y)$.

The moments are now given by the integrals

$$\boxed{M_x = \iint_{\mathcal{R}} y\, \rho(x, y)\, dA, \text{ and } M_y = \iint_{\mathcal{R}} x\rho(x, y)\, dA}$$

Problem 1. Suppose $\mathcal{R}$ is a type I region bounded by $x = a,\ x = b,\ y = 0,$ and $y = f(x)$, with constant density $\rho(x, y) = \rho$. Show that the double-integral definitions of the moments of $\mathcal{R}$ agree with the definitions in equation (1).

The coordinates $(\overline{x}, \overline{y})$ of the *center of mass* of a lamina occupying the region $\mathscr{R}$ and having density function $\rho(x, y)$ are given by

$$\boxed{\overline{x} = \frac{M_y}{m} = \frac{1}{m} \iint_{\mathcal{R}} x\rho(x, y)\, dA, \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{1}{m} \iint_{\mathcal{R}} y\rho(x, y)\, dA}$$

where the mass m of the lamina is given by $m = \iint_{\mathcal{R}} \rho(x, y)\, dA$.

Problem 2. Find the mass and center of mass of the lamina that occupies the triangular region $\mathcal{R}$ with vertices $(0,0)$, $(2,1)$, and $(0,3)$, with density function $\rho(x,y) = x + y$.

Problem 3. Convert the moment and mass integrals to polar coordinates.

Problem 4. A lamina occupies the region inside the circle $x^2 + y^2 = 2y$, but outside the circle $x^2 + y^2 = 1$. Sketch the region. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

Problem 5. Find the mass and center of mass of the cardioid $r = 1 + \cos\theta$ with density function $\rho(x,y) = \sqrt{x^2 + y^2}$.

** Show all work, even if you use Wolfram|Alpha for help. Remember that Wolfram|Alpha, and calculators in general, are *tools* that you use to help you compute. They are not *crutches* that you lean on to be lazy.