Calculus 3: Project 3: Solutions

1. $\left.\begin{array}{l} x = t\cos t \\ y = t\sin t \\ z = t \end{array}\right\}$ $z^2 = t^2$

$x^2+y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2(\cos^2 t+\sin^2 t) = t^2$

thus, $z^2 = x^2+y^2$ for all values of t.

2. $\vec{u} = \langle u_1, u_2, u_3\rangle$

$\vec{v} = \langle v_1, v_2, v_3\rangle$

$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

$$\frac{d}{dt}[\vec{u}\cdot\vec{v}] = \frac{d}{dt}[u_1v_1 + u_2v_2 + u_3v_3]$$
$$= (\dot{u}_1v_1 + u_1\dot{v}_1) + (\dot{u}_2v_2 + u_2\dot{v}_2) + (\dot{u}_3v_3 + u_3\dot{v}_3)$$
$$= (\dot{u}_1v_1 + \dot{u}_2v_2 + \dot{u}_3v_3) + (u_1\dot{v}_1 + u_2\dot{v}_2 + u_3\dot{v}_3)$$
$$= \dot{\vec{u}}\cdot\vec{v} + \vec{u}\cdot\dot{\vec{v}} \quad \blacksquare$$

BONUS 3. $\dfrac{d\vec{T}}{ds} = \dfrac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}$ by the chain rule, and $\dfrac{ds}{dt} = \dfrac{d}{dt}\displaystyle\int_0^t \|\dot{\vec{r}}(u)\|\,du = \|\dot{\vec{r}}(t)\|$ by the FTC.

Thus $\dfrac{d\vec{T}}{ds} = \dfrac{\dot{\vec{T}}(t)}{\|\dot{\vec{r}}(t)\|}$

On the other hand,

$$\kappa(t) = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} \text{ and } \vec{N} = \frac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|}, \text{ so } \kappa\vec{T} = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}\,\frac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|} = \frac{\dot{\vec{T}}}{\|\dot{\vec{r}}\|}$$

Thus $\dfrac{d\vec{T}}{ds} = \kappa\vec{N} \quad \blacksquare$

4. $\left.\begin{array}{l} \|\vec{T}\|^2 = 1 \\ \|\vec{T}\|^2 = \vec{T}\cdot\vec{T} \end{array}\right\}$ $\vec{T}\cdot\vec{T} = 1$ so

$\frac{d}{dt}[\vec{T}\cdot\vec{T} = 1] \Rightarrow \dot{\vec{T}}\cdot\vec{T} + \vec{T}\cdot\dot{\vec{T}} = 0$

or $2\dot{\vec{T}}\cdot\vec{T} = 0$

or $\dot{\vec{T}}\cdot\vec{T} = 0$. Thus $\dot{\vec{T}} \perp \vec{T}$ $\blacksquare$

BONUS 5

$\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$

$\dot{\vec{r}}(t) = \langle 2t, 2t^2, 1 \rangle$

$\|\dot{\vec{r}}(t)\| = \sqrt{4t^2 + 4t^4 + 1} = 2\sqrt{t^4 + t^2 + \frac{1}{4}} = 2\sqrt{(t^2+\frac{1}{2})^2}$

$= 2(t^2 + \frac{1}{2}) = 2t^2 + 1$

so $\vec{T}(t) = \frac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|} = \left\langle \frac{2t}{2t^2+1}, \frac{2t^2}{2t^2+1}, \frac{1}{2t^2+1} \right\rangle$

$\boxed{\vec{T}(1) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle}$

$\dot{\vec{T}}(t) = \left\langle \frac{(2t^2+1)2 - 2t(4t)}{(2t^2+1)^2}, \frac{(2t^2+1)(4t) - 2t^2(4t)}{(2t^2+1)^2}, \frac{-4t}{(2t^2+1)^2} \right\rangle$

$\dot{\vec{T}}(1) = \left\langle \frac{3(2) - 2(4)}{9}, \frac{3(4) - 2(4)}{9}, \frac{-4}{9} \right\rangle = \left\langle \frac{-2}{9}, \frac{4}{9}, \frac{-4}{9} \right\rangle$

$\|\dot{\vec{T}}(1)\| = \frac{1}{9}\sqrt{4+16+16} = \frac{6}{9} = \frac{2}{3}$

so $\boxed{\vec{N}(1) = \frac{\dot{\vec{T}}(1)}{\|\dot{\vec{T}}(1)\|} = \left\langle -\frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \right\rangle}$

and

$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{vmatrix} = \frac{1}{9}\left(-6\vec{i} + 3\vec{j} + 6\vec{k}\right)$

$\boxed{\vec{B}(1) = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle}$

6. $\vec{a}(t) = \langle 2t, \sin t, \cos(2t) \rangle$ $\quad \vec{v}(0) = \langle 1,0,0 \rangle$ $\quad \vec{r}(0) = \langle 0,1,0 \rangle$

$$\vec{v}(t) = \int \vec{a}\, dt = \langle t^2, -\cos t, \tfrac{1}{2}\sin(2t) \rangle + \vec{c}$$
$$\vec{v}(0) = \langle 1,0,0 \rangle = \langle 0,-1,0 \rangle + \vec{c} \Rightarrow \vec{c} = \langle 1,1,0 \rangle$$

$$\boxed{\vec{v}(t) = \langle t^2+1, 1-\cos t, \tfrac{1}{2}\sin(2t) \rangle}$$

$$\vec{r}(t) = \int \vec{v}\, dt = \langle \tfrac{1}{3}t^3 + t, t - \sin t, \tfrac{-1}{4}\cos(2t) \rangle + \vec{c}$$
$$\vec{r}(0) = \langle 0,1,0 \rangle = \langle 0, 0, -\tfrac{1}{4} \rangle + \vec{c} \Rightarrow \vec{c} = \langle 0, 1, \tfrac{1}{4} \rangle$$

so $\boxed{\vec{r}(t) = \langle \tfrac{1}{3}t^3 + t, t - \sin t + 1, \tfrac{1}{4} - \tfrac{1}{4}\cos(2t) \rangle}$

7. a) Write $\vec{v}(t) = \vec{v}(0) - \ln(m(0))\,\vec{v}_e + \ln(m(t))\,\vec{v}_e$

$$\frac{d\vec{v}}{dt} = \frac{m'(t)}{m(t)}\vec{v}_e = \frac{1}{m}\frac{dm}{dt}\vec{v}_e$$

$$\Rightarrow m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{v}_e.$$

b). $\left.\begin{array}{l} \vec{v}(0) = 0 \\ \vec{v}(\tau) = -2\vec{v}_e \end{array}\right\} \Rightarrow +2\cancel{\vec{v}_e} = +\ln\left(\frac{m(0)}{m(\tau)}\right)\cancel{\vec{v}_e}$

$$e^2 = \frac{m(0)}{m(\tau)} \Rightarrow \frac{m(\tau)}{m(0)} = e^{-2}$$