Calculus III: Project 3

Due: Monday, 24 June 2013

Instructions: Complete all problems in a neat and organized fashion on your own paper. If you use Wolfram|Alpha, a calculator, or any other resources, please state what you used it for. You will not lose any points for doing so, as long as you're honest about how and why you used it.

1. Show that the curve with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$, lies on the cone $z^2 = x^2 + y^2$ for all values of t.

2. Prove the *product rule* for the dot product of two vector functions in $\mathbb{R}^3$:

$$\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] = \dot{\mathbf{u}}(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\dot{\mathbf{v}}(t).$$

3. Show that the curvature κ is related to the tangent and normal vectors by the equation

$$\frac{d\,\mathbf{T}}{ds} = \kappa\,\mathbf{N}.$$

4. Prove: $\mathbf{T} \perp \dot{\mathbf{T}}$.

5. Find the tangent, normal, and binormal vectors ($\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$) for the vector function

$$\mathbf{r}(t) = \left\langle t^2,\ \tfrac{2}{3}\,t^3,\ t\right\rangle$$

at the point $(1, \frac{2}{3}, 1)$.

6. Find the position vector of a particle that has the given acceleration and specified initial velocity and initial position.

$$\mathbf{a}(t) = \langle 2t,\ \sin t,\ \cos(2t)\rangle\,,\quad \mathbf{v}(0) = \mathbf{i},\quad \mathbf{r}(0) = \mathbf{j}.$$

7. A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t. If the exhaust gases escape with velocity $\mathbf{v}_e$ relative to the rocket, it can be deduced from Newton's second law of motion that

$$m\,\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e.$$

a) Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln\left(\frac{m(0)}{m(t)}\right)\mathbf{v}_e$.

b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases ($2\,\mathbf{v}_e$), what fraction of its initial mass would the rocket have to burn as fuel?