Calc III : Project 2 : Solutions

1. Complete a few squares

$$x^2 + 8x + 16 + y^2 - 6y + 9 + z^2 + 2z + 1 = -17 + 16 + 9 + 1$$

$$\Rightarrow (x+4)^2 + (y-3)^2 + (z+1)^2 = 9$$

Thus center : $C(-4, 3, -1)$
radius : $r = 3$

2. Complete a few squares on the second guy:

$$x^2 - 8x + 16 + y^2 - 4y + 4 + z^2 - 8z + 16 = -11 + 16 + 4 + 16$$

$$(x-4)^2 + (y-2)^2 + (z-4)^2 = 25$$

center is $(4,2,4)$
distance between centers is $\sqrt{4^2+2^2+4^2} = \sqrt{36} = 6$.

3. $\frac{dy}{dx} = 2x$, slope at $(2,4)$ is $2(2) = 4$

Therefore, the direction of the tangent vectors is $\langle 1, 4 \rangle$
The length of this is $\sqrt{1^2+4^2} = \sqrt{17}$
So the unit vectors are:

$$\left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle \text{ and } \left\langle \frac{-1}{\sqrt{17}}, \frac{-4}{\sqrt{17}} \right\rangle.$$

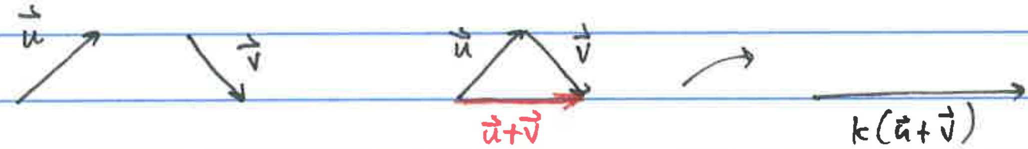
RB

4\. a) $\vec{u} = \langle u_1, u_2, u_3 \rangle \qquad \vec{v} = \langle v_1, v_2, v_3 \rangle$ algebraically

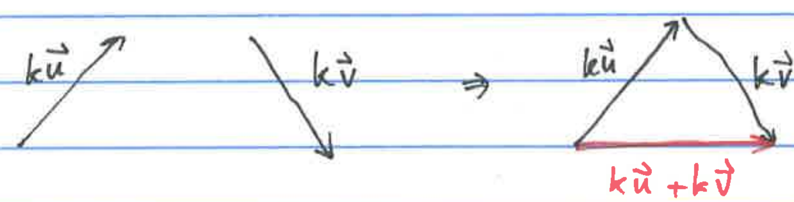
$$\vec{u}+\vec{v} = \langle u_1+v_1,\ u_2+v_2,\ u_3+v_3 \rangle$$

$$\begin{aligned} k(\vec{u}+\vec{v}) &= k\langle u_1+v_1,\ u_2+v_2,\ u_3+v_3 \rangle \\ &= \langle k(u_1+v_1),\ k(u_2+v_2),\ k(u_3+v_3) \rangle \\ &= \langle ku_1+kv_1,\ ku_2+kv_2,\ ku_3+kv_3 \rangle \\ &= \langle ku_1, ku_2, ku_3 \rangle + \langle kv_1, kv_2, kv_3 \rangle \\ &= k\langle u_1, u_2, u_3 \rangle + k\langle v_1, v_2, v_3 \rangle \\ &= k\vec{u} + k\vec{v} \quad \blacksquare \end{aligned}$$

b)

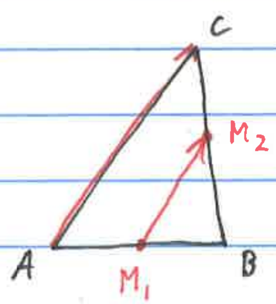


multiplying by k ~~magnitudes~~ stretches the vector by a factor of k.



Since the ∡s are the same, the sides must be proportional. $\blacksquare$

5\.



If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ $\Big\}$ $M_1 = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, $M_2 = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$

Then $\vec{AC} = \langle x_3-x_1,\ y_3-y_1 \rangle$ and

$$\begin{aligned} \vec{M_1M_2} &= \left\langle \frac{x_2+x_3}{2} - \frac{x_1+x_2}{2},\ \frac{y_2+y_3}{2} - \frac{y_1+y_2}{2} \right\rangle \\ &= \left\langle \tfrac{1}{2}(x_3-x_1),\ \tfrac{1}{2}(y_3-y_1) \right\rangle \\ &= \tfrac{1}{2}\vec{AC} \quad \blacksquare \end{aligned}$$

6. $|\vec{u}\cdot\vec{v}| = \left|\, \|\vec{u}\|\,\|\vec{v}\|\cos\theta \right|$

$= \|\vec{u}\|\,\|\vec{v}\|\,|\cos\theta|$

but $|\cos\theta| \leq 1$, so

$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$ ∎

This is equal only when $\cos\theta = \pm 1$, which happens only when $\theta = 0$ or π.

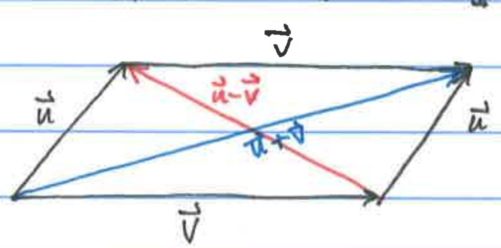
This means that $|\vec{u}\cdot\vec{v}| = \|\vec{u}\|\,\|\vec{v}\|$ only if $\vec{u} \parallel \vec{v}$.

7. a) Geometrically this means that the length of the hypotenuse of a triangle is less than or equal to the length of the sum of the sides.

b) Prove this instead: $\|\vec{u}+\vec{v}\|^2 \leq (\|u\|+\|v\|)^2$.

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \\
&\leq \|\vec{u}\|^2 + 2\,|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2 \\
&\leq \|\vec{u}\|^2 + 2\,\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2
\end{aligned}$$ ∎

8. a)



The Parallelogram Law is a generalization of the Pythagorean Theorem: the sum of the squares of the sides is equal to the sum of the squares of the diagonals. ∎

8. b) From problem 7: $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$

Similarly, $+\ \|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$

Add them up: $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$ ■

9. $(\vec{u}+\vec{v}) \perp (\vec{u}-\vec{v}) \Rightarrow (\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$

but $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v}$

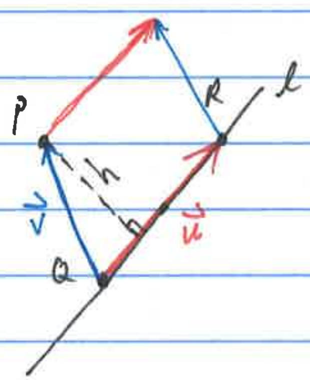
$= \|\vec{u}\|^2 - \|\vec{v}\|^2$

which implies $\|\vec{u}\|^2 = \|\vec{v}\|^2$

since they're both positive (they're lengths!):

$\|\vec{u}\| = \|\vec{v}\|$ ■

10. a)



h = height of parallelogram

h = distance from P to l also.

Area of parallelogram $= \|\vec{u}\| \cdot h$

(base × height)

Area of parallelogram also $= \|\vec{u}\times\vec{v}\|$

Thus $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\cdot h$

$\Rightarrow h = \dfrac{\|\vec{u}\times\vec{v}\|}{\|\vec{u}\|}$ ■

b) $\vec{QR} = \langle -1,-2,-1\rangle$ $\quad \vec{QP} = \langle 1,-5,-7\rangle$

$\|\vec{QR}\| = \sqrt{1+4+1} = \sqrt{6}$

$\vec{QR}\times\vec{QP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & -1 \\ 1 & -5 & -7 \end{vmatrix} = \vec{i}(9) - \vec{j}(8) + \vec{k}(7)$

$= \langle 9,-8,7\rangle$

$\|\vec{QR}\times\vec{QP}\| = \sqrt{81+64+49} = \sqrt{184}$

so $h = \sqrt{\dfrac{184}{6}} = \boxed{\sqrt{\dfrac{92}{3}}} \approx 5.538$ (via Wolfram|Alpha) ■