Calculus III: Project 2

Due: Monday, 17 June 2013

Instructions: Complete all problems in a neat and organized fashion on your own paper. If you use Wolfram|Alpha, a calculator, or any other resources, please state what you used it for. You will not lose any points for doing so, as long as you're honest about how and why you used it.

1. Show that the equation represents a sphere. Find its center and radius.

$$x^2 + y^2 + z^2 + 17 = 6y - 8x - 2z$$

2. Find the distance between the centers of the spheres

$$x^2 + y^2 + z^2 = 4, \quad \text{and}$$

$$x^2 + y^2 + z^2 = 8x + 4y + 8z - 11.$$

3. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

4. Let $k \in \mathbb{R}$. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^3$, then scalar multiplication distributes over vector addition:

$$k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}.$$

(a) Prove this property algebraically, then (b) use similar triangles to give a geometric proof.

5. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

6. Use the fact that $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta$ to prove the *Cauchy-Schwarz Inequality*:

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|.$$

When is this actually "="?

7. The *Triangle Inequality* for vectors is

$$\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|.$$

(a) Give a geometric interpretation of the Triangle Inequality.

(b) Use the Cauchy-Schwarz Inequality to prove the triangle inequality.
[Hint: $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\boldsymbol{u}+\boldsymbol{v}) \cdot (\boldsymbol{u}+\boldsymbol{v})$]

8. The *Parallelogram Law* states that

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 + \|\boldsymbol{u}-\boldsymbol{v}\|^2 = 2\,\|\boldsymbol{u}\|^2 + 2\,\|\boldsymbol{v}\|^2.$$

(a) Give a geometric interpretation of the Parallelogram Law.

(b) Prove the Parallelogram Law.
[Hint: Use the hint from the last problem.]

9. Show that if $\boldsymbol{u}+\boldsymbol{v}$ and $\boldsymbol{u}-\boldsymbol{v}$ are orthogonal, then $\|\boldsymbol{u}\| = \|\boldsymbol{v}\|$.

10. Let P be a point not on the line ℓ that passes through the points Q and R.
(a) Show that the (shortest) distance d from the point P to the line ℓ is

$$d = \frac{\|\boldsymbol{u} \times \boldsymbol{v}\|}{\|\boldsymbol{u}\|}$$

where $\boldsymbol{u} = \vec{QR}$ and $\boldsymbol{v} = \vec{QP}$.

(b) Find the distance between the point $P(1,1,1)$ and the line determined by $Q(0,6,8)$ and $R(-1,4,7)$.