Chapter 13: Vector Calculus

i.e., "the calculus of vector fields"

Question: What is a vector field?

One answer: If $\mathbb{V}^n$ is the vector space associated to (equivalent to) $\mathbb{R}^n$, then a vector field $\vec{F}$ is a function $\vec{F}: \mathbb{R}^n \to \mathbb{V}^n$ that takes points in $\mathbb{R}^n$ as arguments, and has n-vectors as outputs.

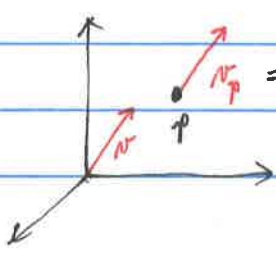
or $D \subset \mathbb{R}^n$ some domain.

This would be called an n-vector field. We are interested in the cases when $n=2$ or 3.

This answer is probably good enough to do the calculus, but it lacks some intuition. For this reason, we digress from the book for a moment and look at vector fields in a slightly different ilk.

Recall that all vectors in $\mathbb{V}^n$ have the origin as their initial pt. We ~~can think of these~~ want to move the initial pt to some other point $p \in \mathbb{R}^n$. To do this, we think of having a copy of $\mathbb{R}^n$ and a copy of $\mathbb{V}^n$ (that just looks like another copy of $\mathbb{R}^n$, but thought of as vectors). For any $p \in \mathbb{R}^n$ and $v \in \mathbb{V}^n$ we can form the pointed vector v_p:

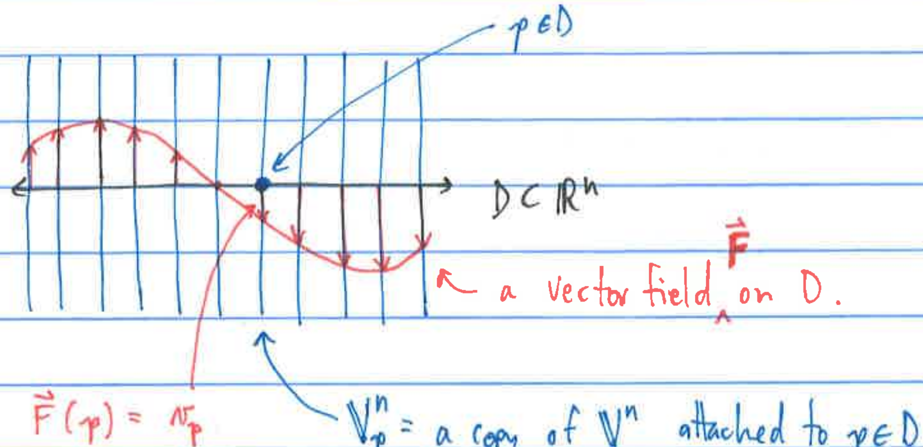
drawn in $\mathbb{R}^3 = \mathbb{V}^3$:



$v_p = v$ translated to p.

We write $\mathbb{V}_p^n$ to denote the pointed vector space $\mathbb{V}^n$ w/ initial point $p \in \mathbb{R}^n$ (it still just looks like a copy of $\mathbb{R}^n$).

Now for any domain $D \subset \mathbb{R}^n$, think of "attaching" the pointed vector spaces $\mathbb{V}_p^n$ to every $p \in D$. We can draw a schematic picture:

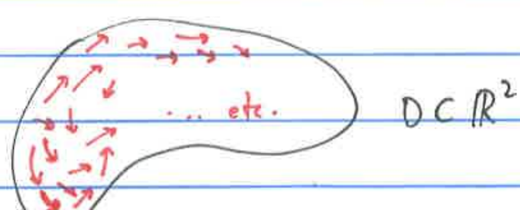


caveat: this is a "smooth" vector field. Any arbitrary v.f. might not make such a "nice looking" curve.

Think of $\mathbb{R}^n$ and $\mathbb{V}^n$ as just $\mathbb{R}^1$ and $\mathbb{V}^1$ in this picture.

Now, a vector field on D picks out one vector from each $\mathbb{V}_p$, for all $p \in D$.
This is really just a different way of thinking of our first definition. The function $\vec{F}$ picks out exactly one $v \in \mathbb{V}^n$ for each $p \in D$ and "glues" it to p. (Thus making it look like a vector in v_p, so why not just start there?).

In $\mathbb{R}^2$ we can envision this as arrows attached to the plane:



Real applications of vector fields are: velocity fields (wind, water), force fields (gravity, electromagnetism, etc.).

Look in the book for good pictures (I can't draw).
There are some nice pics of 3D vector fields.

Examples of vector fields

1.) $\vec{F}(x,y) = -y\vec{i} + x\vec{j} = \begin{pmatrix} -y \\ x \end{pmatrix} = \langle -y, x \rangle$

* Plug in some points and plot some vectors.

e.g., $(1,0), (0,1), (-1,0), (0,-1), \quad (2,2), \quad (0,3)$

The vectors appear to be tangent to circles centered at the origin. Take the dot product w/ the position vector $\langle x,y\rangle = \vec{x}$

so, $\vec{x} \cdot \vec{F}(\vec{x}) = \langle x,y\rangle \cdot \langle -y, x\rangle = -xy + xy = 0$
This shows that $\vec{x} \perp \vec{F}(\vec{x})$ for any $\vec{x} \in \mathbb{R}^2 (= \mathbb{V}^2)$.
Thus it is tangent to the circle w/ center $\vec{0}$ and radius $\|\vec{x}\|$.
Also, notice that $\|\vec{F}(\vec{x})\| = r = \|\vec{x}\|$.

2.) $\vec{F}(x,y,z) = z\vec{k} = \langle 0,0,z\rangle$

These look like "vertical" vectors that get longer as we move further from the xy-plane.

3.) A fluid is flowing steadily through a pipe in $\mathbb{R}^3$. (this is called Couette Flow (sic?).) Let a vector $v(x,y,z) = v_p$ represent the velocity of a "parcel" of fluid in the flow at a point p. The speed of the flow at any point is given by the length of the velocity vector.

4.) Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects w/ masses m and M is

$$\|\vec{F}(r)\| = \frac{mMG}{r^2} \quad \text{(the well known inverse square law)}$$

where r is the distance between the centers.

If $\vec{x}$ is a position vector in $\mathbb{R}^3$, $\vec{x} = \langle x, y, z \rangle$, representing the center of one object, where the other is centered at the origin, then

$$r = \|\vec{x}\| \quad \text{and}$$

$$r^2 = \|\vec{x}\|^2 = x^2 + y^2 + z^2$$

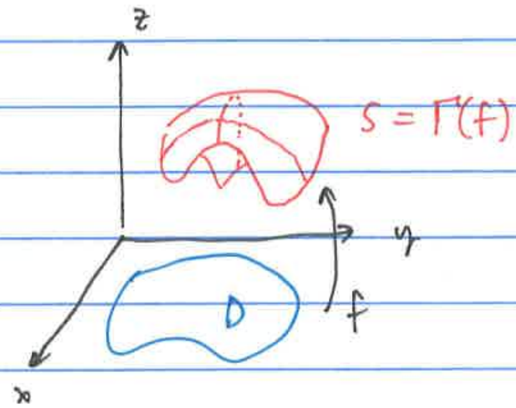
The gravitational force is exerted on the second force, pointing toward the origin, so the direction is the same as the unit vector $-\vec{x}/\|\vec{x}\| = \vec{u}$

Thus, the gravitational force is given by

$$\vec{F}(\vec{x}) = \|\vec{F}(\vec{x})\|\,\vec{u} = \frac{mMG}{\|\vec{x}\|^2}\left(\frac{-\vec{x}}{\|\vec{x}\|}\right) = -\frac{mMG}{\|\vec{x}\|^3}\vec{x}$$

Gradient Fields

We've already seen one of the most important examples of a vector field. If f is a scalar function of two variables, then f determines a surface in $\mathbb{R}^3$.



There is a natural vector field on $\Gamma(f)$ called the gradient vector field of f.

It is the vector field that satisfies

$$D_{\vec{u}} f(\vec{x}) = \left[\nabla f(\vec{x})\right] \cdot \vec{u}$$

for all unit $\vec{u} \in \mathbb{R}^2$ (or $\mathbb{V}^2$).

Recall that $\nabla f = \langle D_x f, D_y f \rangle$ in 2-dimensions, and
$\nabla f = \langle D_x f, D_y f, D_z f \rangle$ in 3-d.

* (these are all functions of $\vec{x}$)

Furthermore, $\nabla f(\vec{x})$ is the vector in the direction of "steepest" ascent at $\vec{x}$ of the graph $\Gamma(f)$.

i.e., f is changing the most in the direction of $\nabla f(\vec{x})$ (at $\vec{x}$.)

The most important ~~fact~~ definition that we need from this section is:

Defn. A vector field $\vec{F}$ is called a conservative vector field if it is the gradient of some scalar function f.

$$\text{i.e., } \quad \vec{F}(\vec{x}) = \nabla f(\vec{x})$$

for all $\vec{x} \in D$, and some $f : D \to \mathbb{R}$.

If ~~this~~ $\vec{F}$ is a conservative v.f., then the function $f : D \to \mathbb{R}$ is called a potential function for $\vec{F}$.

Not all v.f.s are conservative, but most v.f.s in physics are (hence the name).

Ex. The gravitational field is conservative. Verify this by putting

$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}} .$$