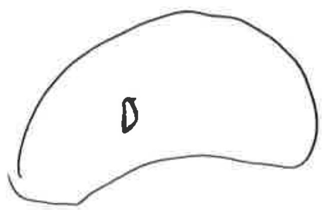
§ 12.4: Some applications

In Chapter 7 we used single integrals to compute moments and centers of mass of a thin plate with constant density.

Now suppose density is a function of x any y; $\rho = \rho(x,y)$

$$\rho(x,y) = \lim \frac{\Delta m}{\Delta A}$$ where Δm and ΔA are the mass and area of a small rectangle that contains (x,y).

To find the total mass of a plate



We divide it up into a bunch of small rectangles and assume $\rho = 0$ outside of D. Using a Riemann sum type argument, then

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

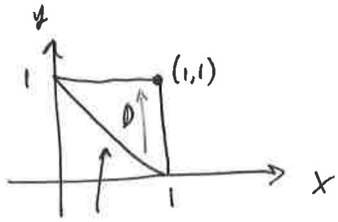
then in the limit we get

$$\boxed{m = \iint_D \rho(x,y)\, dA}$$

Another example is an electric charge Q w/ charge density $\sigma(x,y)$:

$$\boxed{Q = \iint_D \sigma(x,y)\, dA}$$

Ex. Charge is distributed over the triangular region D:



Charge density is given by $\sigma(x,y) = xy$. C/m^2.
Find the total charge Q.

$$Q = \iint_D xy\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx$$

$$= \int_0^1 \tfrac{1}{2}xy^2 \Big|_{1-x}^{1} dx$$

$$= \int_0^1 \tfrac{1}{2}x - \tfrac{1}{2}x(1-x)^2\, dx$$

$$= \int_0^1 \tfrac{1}{2}x - \tfrac{1}{2}x(1-2x+x^2)\, dx$$

$$= \int_0^1 \cancel{\tfrac{1}{2}x} - \cancel{\tfrac{1}{2}x} + x^2 - \tfrac{1}{2}x^3\, dx$$

$$= \int_0^1 x^2 - \tfrac{1}{2}x^3\, dx$$

$$= \tfrac{1}{3}x^3 - \tfrac{1}{8}x^4 \Big|_0^1 = \tfrac{1}{3} - \tfrac{1}{8} = \boxed{\tfrac{5}{24}\ C}$$

<u>Moments and Centers of Mass</u> can be defined similarly:

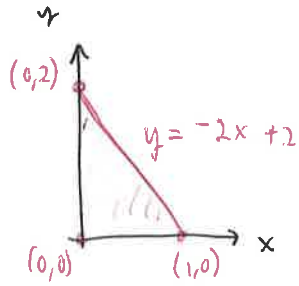
<u>the moments:</u> $M_x = \iint_D y\, \rho(x,y)\, dA$

$$M_y = \iint_D x\, \rho(x,y)\, dA$$

<u>Center of mass:</u>

$$\left.\begin{aligned} \bar{x} &= \frac{M_y}{m} = \ = \frac{1}{m}\iint_D x\, \rho(x,y)\, dA \\ \bar{y} &= \frac{M_x}{m} = \frac{1}{m}\iint_D y\, \rho(x,y)\, dA \end{aligned}\right\} \quad m = \iint_D \rho(x,y)\, dA$$

Ex. Find the mass and center of mass of a triangular region w/ vertices $(0,0)$, $(1,0)$, $(0,2)$ $\rho(x,y) = 1 + 3x + \text{[redacted]}y$.



$$m = \iint_D \rho(x,y)\, dA$$

$$= \int_0^1 \int_0^{-2x+2} 1 + 3x + \text{[redacted]}y \; dy\, dx$$

$$= \int_0^1 y + 3xy + \frac{y^2}{2} \Big|_0^{-2x+2} dx$$

$$= \int_0^1 -2x+2 + 3x(-2x+2) + \frac{(2-2x)^2}{2} \, dx$$

$$= \int_0^1 -2x + 2 - 6x^2 + 6x + \frac{(4 - 8x + 4x^2)}{2} \, dx$$

$$\cancel{= \int_0^1 6 - 4x - 2x^2 \, dx}$$

$$= \cancel{6x - 2x^2 - \tfrac{2}{3}x^3 \Big|_0^1 = 6 - 2 - \tfrac{2}{3} = \tfrac{10}{3}}$$

$$= \int_0^1 -2x + 2 - 6x^2 + 6x + 2 - 4x + 2x^2 \, dx$$

$$= \int_0^1 4 - 4x^2 \, dx = 4x - \frac{4}{3}x^3 \Big|_0^1 = 4 - \frac{4}{3} = \boxed{\frac{8}{3}}$$

Center of mass: FTIS.