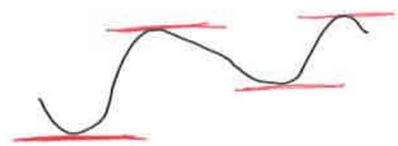
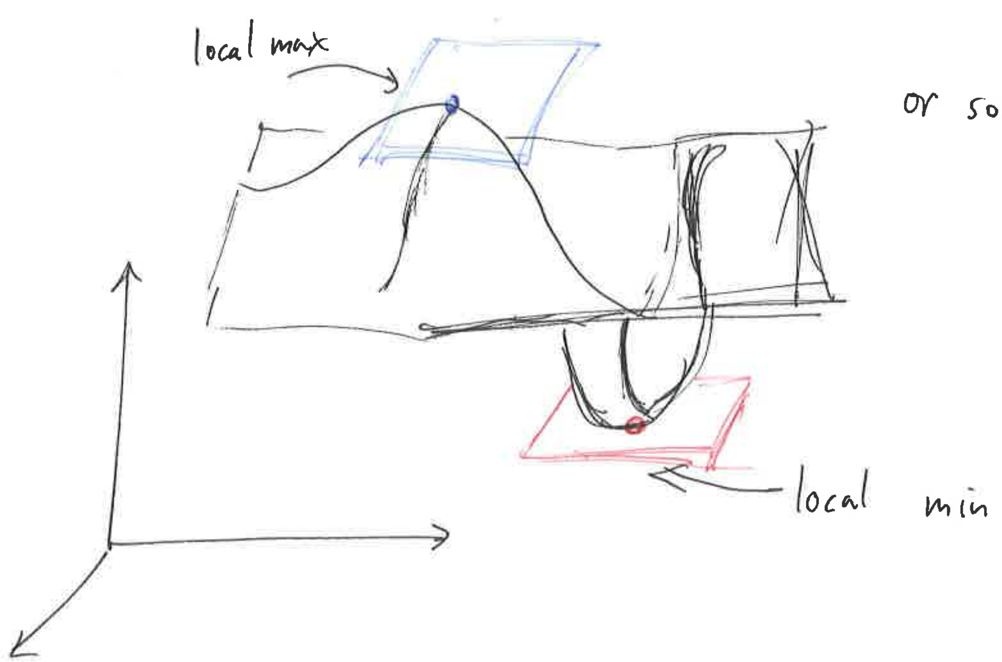
11.7: Maximum and minimum values

In Calc I, when a differentiable function had a local max or min, then its derivative equaled 0 at that point: the tangent line was horizontal.



... etc.

In Calc III, if a surface has a local max or min, then its tangent plane at the point is horizontal (parallel to the domain in the xy-plane).



or something...

Defn. A function of two variables has a <u>local maximum</u> at (a,b) if $f(x,y) \le f(a,b)$ for all (x,y) in a neighborhood of (a,b).

similarly: local minimum $\ge$

The number $f(a,b)$ is the <u>local maximum (or minimum) value</u> of f.

Thm. If f has a local extremum at (a,b) and the first order partial derivatives exist there, then $D_x f(a,b) = D_y f(a,b) = 0$.

The point (a,b) is called a critical point (or stationary point) of f if $D_x f(a,b) = 0$ & $D_y f(a,b) = 0$, or if one (or both) do not exist.

Ex. Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

Then
$$D_x f = 2x - 2$$
$$D_y f = 2y - 6$$

These $= 0$ when $x = 1$, $y = 3$. So the only critical point is $(1,3)$.

Is this a max, min, or neither?

Complete a couple of squares to get:

$$f(x,y) = 4 + (x-1)^2 + (y-3)^2$$

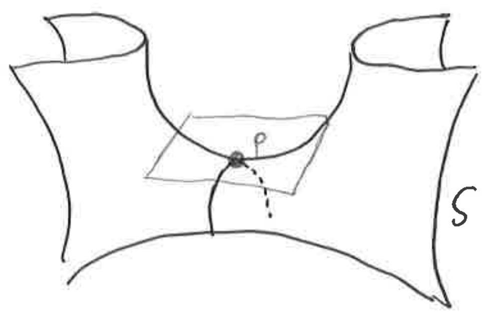
This is a paraboloid that opens up. So the point $(1, 3, f(1,3))$ is a minimum.

$f(1,3) = 4 + (0)^2 + (0)^2 = 4$ is the local minimum value.

Ex. Find the extreme values of $f(x,y) = y^2 - x^2$

$$\left.\begin{array}{l} D_x f = -2x \\ D_y f = 2y \end{array}\right\} \quad \text{critical point at } (0,0)$$

Recalling §10.6, this is a hyperbolic paraboloid, or a saddle surface. The point $(0,0)$ is the saddle point.



T_pS is half above and half below the surface at P. Therefore $f(0,0)$ is not a max or a min.

Thm. The Second Derivative Test

Suppose the second partial derivatives of f are continuous near a critical point (a,b). Define

$$H = H(a,b) = D_{xx}f(a,b)\, D_{yy}f(a,b) - [D_{xy}f(a,b)]^2$$

a) If $H > 0$ and $D_{xx}f(a,b) > 0$, then $f(a,b)$ is a local min.

b) If $H > 0$ and $D_{xx}f(a,b) < 0$, then $f(a,b)$ is a local max.

c) If $H < 0$, then $f(a,b)$ is neither a max nor a min; i.e., (a,b) is a saddle point.

Notes: 1. If $H = 0$, then we learn nothing.

2. The matrix $\begin{bmatrix} D_{xx}f & D_{xy}f \\ D_{yx}f & D_{yy}f \end{bmatrix}$ is called the Hessian matrix of f.

This H is its determinant, $H = \begin{vmatrix} D_{xx}f & D_{xy}f \\ D_{yx}f & D_{yy}f \end{vmatrix} = D_{xx}f\, D_{yy}f - (D_{xy}f)^2$.

Ex. Find and classify all critical points of $f(x,y) = x^4 + y^4 - 4xy + 1$

$$\begin{aligned} D_x f &= 4x^3 - 4y \\ D_y f &= 4y^3 - 4x \end{aligned} \Rightarrow \begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \Rightarrow \begin{aligned} y &= x^3 \\ y^3 &= x \end{aligned}$$

$$\Rightarrow (x^3)^3 - x = 0$$

$$\Rightarrow x(x^8 - 1) = 0 \qquad x = 0, +1, -1$$

So the three critical points are $(0,0)$, $(1,1)$, $(-1,-1)$.

Second derivatives:

$$D_{xx} f = 12x^2 \qquad D_{xy} f = -4$$

$$D_{yy} f = 12y^2$$

$$H = 144x^2y^2 - 16$$

$H(0,0) = -16 < 0 \Rightarrow (0,0)$ is a saddle point

$H(1,1) = H(-1,-1) = 128 > 0$ so $(1,1)$ and $(-1,-1)$ are local min.

since $D_{xx} f(1,1) = D_{xx} f(-1,-1) = 12 > 0$.

Absolute Extrema

Thm. If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum and absolute minimum value.

To find absolute extrema:

1. Find the values of f at critical pts. (critical values)
2. Find the extreme values of f on the boundary of D
3. The largest and smallest of these values are the max and min, respectively.

$\underline{\text{Ex}}$. $f(x,y) = x^2 - 2xy + 2y$ on $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

$D_x f = 2x - 2y$

$D_y f = -2x + 2$ $\Rightarrow$ $x = 1$, $y = 1$ so crit pt. is $(1,1)$

$f(1,1) = 1 - 2 + 2 = 1$.

when

$x=0$: $f(0,y) = 2y$ has min 0 and max 4

$x=3$: $f(3,y) = 9 - 6y + 2y$

$= 9 - 4y$ has max 9 and min 1

$y=0$: $f(x,0) = x^2$ has min 0 and max 9

$y=3$: $f(x,3) = x^2 - 4x + 4 = (x-2)^2$ has min 0 and max 4

So the absolute max of f on D is 9

and absolute min of f on D is 0.