§11.6: Directional Derivatives and the gradient vector field

Recall that

$$D_x f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

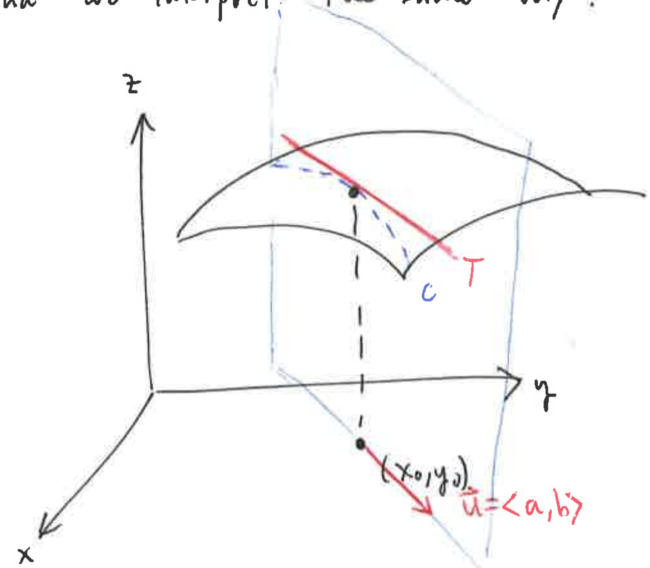
is the derivative of f in the x-direction; i.e., the $\vec{i}$ direction.

Similarly, $D_y f(x_0,y_0)$ is the derivative of f in the direction of the vector $\vec{j}$ at the point (x_0,y_0).

where $\vec{i} = \langle 1,0\rangle$ and $\vec{j} = \langle 0,1\rangle$.

Question: Can we take the derivative of f at (x_0,y_0) in the direction of a vector $\vec{u} = \langle a,b\rangle$?

Yes and we interpret the same way:



$\vec{u}$ is a unit vector $\|\vec{u}\|=1$.

The directional derivative of f at (x_0,y_0) in the direction of a unit vector $\vec{u}=\langle a,b\rangle$ is the slope of the tangent line T above, given by

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h}$$

if the limit exists.

Ex. Verify that $D_x f = D_{\vec{i}} f$ and $D_y f = D_{\vec{j}} f$. by this definition.

To compute directional derivatives we ~~compute~~ use:

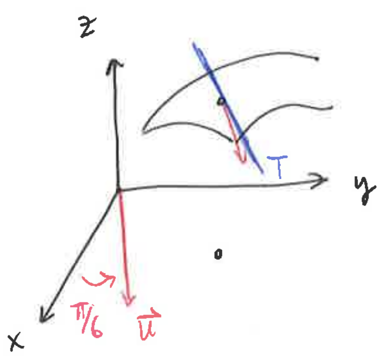
Theorem. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$\boxed{D_{\vec{u}} f(x,y) = D_x f(x,y) \cdot a + D_y f(x,y) \cdot b}$$

Proof. RE. $\square$

Ex. $f(x,y) = x^3 - 3xy + 4y^2$

$\vec{u}$ is the unit vector that forms a $\pi/6$ angle w/ the positive x-axis, at the point $(1,2)$



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So $\vec{u} = \langle \cos \pi/6, \sin \pi/6 \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$D_x f = 3x^2 - 3y \qquad D_x f(1,2) = 3 - 6 = -3$

$D_y f = -3x + 8y \qquad D_y f(1,2) = -3 + 16 = 13$

Then $D_{\vec{u}} f(1,2) = (-3)\left(\frac{\sqrt{3}}{2}\right) + (13)\left(\frac{1}{2}\right) = \boxed{\frac{-3\sqrt{3} + 13}{2}}$

The Gradient Vector

Notice from the theorem:

$$\begin{aligned} D_{\vec{u}} f &= D_x f \cdot a + D_y f \cdot b \\ &= \langle D_x f, D_y f \rangle \cdot \langle a, b \rangle \\ &= \langle D_x f, D_y f \rangle \cdot \vec{u} \end{aligned}$$

So the directional derivative of f is some vector dotted with the unit direction vector.

Def'n. If f is a function of x and y, then the gradient of f is the vector function grad $f = \nabla f$ defined by

$$\boxed{\nabla f(x,y) = \langle D_x f(x,y), D_y f(x,y) \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}}$$

In this notation we can rewrite the formula for directional derivatives as

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}}$$

Ex. $f(x,y) = x^2 y^3 - 4y$, compute $D_{\vec{v}} f(2,-1)$ for $\vec{v} = \langle 2,5 \rangle$

$$\nabla f(x,y) = \langle 2xy^3, 3x^2y^2 - 4 \rangle$$

$$\nabla f(2,-1) = \langle -4, 8 \rangle$$

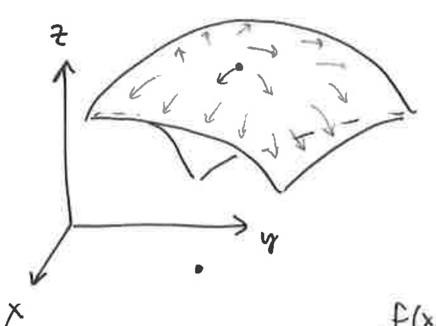
$$D_{\vec{v}} f(2,-1) = \frac{\langle -4, 8 \rangle \cdot \langle 2,5 \rangle}{\| \langle 2,5 \rangle \|} = \frac{-8+40}{\sqrt{4+25}} = \boxed{\frac{32}{\sqrt{29}}}$$

We write the gradient vector field as

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

For every point in the domain D of f, this assigns a gradient vector to the image point in the surface $S = \Gamma(f)$.

The vector points in the direction of steepest ascent or descent.



∇f as a vector field.

Given a point $p(x,y) \in D$, then $(x, y, f(x,y)) =: P \in \Gamma(f) = S$,

~~Let $T_P(p)$ be the space of all pointed vectors~~

There is a unique vector $\vec{g}_p \in T_P S$ such that

$$D_{\vec{u}} f = \vec{g}_p \cdot \vec{u} \quad \text{for all } \vec{u} \in D.$$

This is the gradient vector of f at p : $\vec{g}_p = \nabla f(p) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

This is the geometric intuitive idea.

<u>Ex</u>. $f(x,y) = 1 + 2x\sqrt{y}$, $p(3,4)$, $\vec{v} = \langle 4, -3 \rangle$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 4, -3 \rangle}{5} = \langle 4/5, -3/5 \rangle$$

$D_x f = 2\sqrt{y}$ $\quad D_x f(3,4) = 2\sqrt{4} = 4$

$D_y f = \frac{x}{\sqrt{y}}$ $\quad D_y f(3,4) = \frac{3}{2}$

$$D_{\vec{u}} f(3,4) = \frac{4}{5}(4) + \left(\frac{-3}{5}\right)\left(\frac{3}{2}\right)$$

$$= \frac{16}{5} - \frac{9}{10} = \frac{23}{10} \checkmark$$

Functions of three variables

Def'n. The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = \langle a, b, c\rangle$ is

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh, z_0+ch) - f(x_0, y_0, z_0)}{h}$$

if the limit exists.

In vector notation this can be written as

$$\boxed{D_u f(\vec{x}_0) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}}$$

or $\boxed{\frac{d}{dt} f(\vec{x} + t\vec{u})\Big|_{t=0}.}$

Again this can be represented as a dot product w/ the gradient vector at $\vec{x}_0$:

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

where now the gradient vector field is given by

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

Ex. $f(x,y,z) = x\sin(yz)$

a.) Find ∇f

b.) Find $D_{\vec{u}} f$ for $\vec{v} = \langle 1, 2, -1\rangle$ at $(1,3,0) = p$.

$$\nabla f = \langle \sin(yz),\ xz\cos(yz),\ xy\cos(yz)\rangle$$

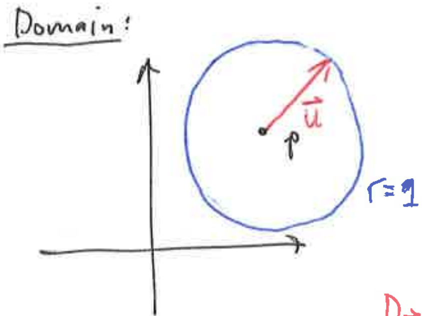
$$\nabla f(p) = \langle 0,\ 0,\ 3\rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1,2,-1\rangle}{\sqrt{6}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\rangle$$

$$D_{\vec{u}} f = 0 + 0 - \frac{3}{\sqrt{6}} = -\sqrt{\frac{3}{2}}.$$

Maximizing the directional derivative

Suppose we take all directional derivatives of a function at a point p:



"sweep through" all directions.
What can we say about the direction whose derivative is a maximum?

$D_{\vec{u}} f$ = maximum

Thm. Suppose f is a differentiable function of 2, 3, (or more) variables. The maximum value of the directional derivative $D_{\vec{u}} f(x)$ is $\|\nabla f(x)\|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector at x, $\nabla f(x)$.

In other words (as we've said before) the gradient vector of f is the direction of steepest ascent.

Pf. $D_{\vec{u}} f = \nabla f \cdot u = \|\nabla f\| \|u\| \cos\theta = \|\nabla f\| \cos\theta$

by a dot product identity from Ch. 10. The maximum value of cosine is 1 and this occurs when $\theta = 0$.

Therefore $\max\{D_{\vec{u}} f\} = \|\nabla f\|$ and it occurs when $\theta = 0$, hence $\vec{u}$ and ∇f have the same direction.

Ex. a) $f(x,y) = xe^y$. Find the rate of change of f at $P(2,0)$ in the direction toward $Q(\frac{1}{2},2)$

$$\vec{v} = \langle -\tfrac{3}{2}, 2 \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -3/2, 2 \rangle}{\sqrt{9/4 + 4}} = \frac{\langle -3/2, 2 \rangle}{5/2} = \langle -3/5, 4/5 \rangle$$

$\nabla f = \langle e^y, xe^y \rangle$, $\nabla f(P) = \langle 1, 2 \rangle$

So $D_{\vec{u}} f = -3/5(1) + 4/5(2) = \frac{-3}{5} + \frac{8}{5} = \frac{5}{5} = 1$

b) In what direction does f have maximum rate of change? what is it?

max rate of change is at $\nabla f(P) = \langle 1, 2 \rangle$

and rate is $\|\langle 1,2 \rangle\| = \sqrt{5}$

Ex. (16) Find the maximum rate of change of f at the point:

$f(p,q) = qe^{-p} + pe^{-q}$ at $(0,0)$

$\nabla f(p,q) = \langle -qe^{-p} + e^{-q}, e^{-p} - pe^{-q} \rangle$

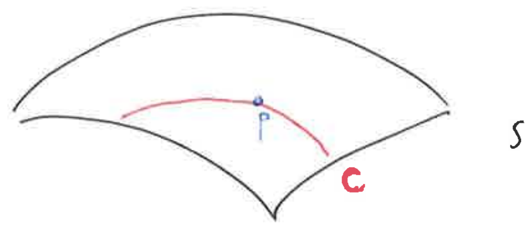
$\nabla f(0,0) = \langle 1, 1 \rangle$

$\|\nabla f(0,0)\| = \sqrt{2}$

Tangent Planes to level surfaces

Suppose $F(x,y,z)$ is a function of 3 variables, such that the level surface $F(x,y,z)=k$ is a surface S in $\mathbb{R}^3$.

Let C be any curve in S that passes through $P=(x_0,y_0,z_0)$



The curve C can be described by a vector function

$$\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$$

Suppose $t=0$ corresponds to P; i.e., $\vec{r}(0)=\vec{r}_0=\langle x(0),y(0),z(0)\rangle=\vec{OP}$.

Since C is in S, then any point on C must satisfy

$$F(x(t),y(t),z(t))=k$$

If x,y,z are differentiable in t, then by the chain rule we have:

$$\underline{\frac{\partial F}{\partial x}}\,\underline{\frac{dx}{dt}}+\underline{\frac{\partial F}{\partial y}}\,\underline{\frac{dy}{dt}}+\underline{\frac{\partial F}{\partial z}}\,\underline{\frac{dz}{dt}}=0$$

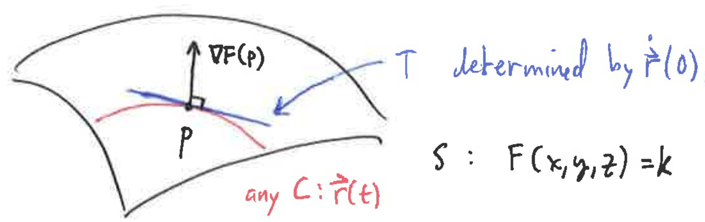
or $$\underline{\nabla F\cdot\dot{\vec{r}}=0}$$

In particular we have $\vec{r}(0) = \langle x_0, y_0, z_0 \rangle$ so that

$$\nabla F(x_0, y_0, z_0) \cdot \dot{\vec{r}}(0) = 0$$

This means that the gradient vector to F at P is orthogonal (or perpendicular) to the tangent vector $\dot{\vec{r}}(0)$ of <u>any</u> curve C at P.

Furthermore ∇F is the normal vector to its level surfaces at any point:



Now we can write an equation of the tangent plane to S at P:

T_pS: $\quad n \cdot (\vec{r} - \vec{r}_0) = 0$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\Rightarrow \boxed{D_x F(x_0, y_0, z_0)(x - x_0) + D_y F(x_0, y_0, z_0)(y - y_0) + D_z F(x_0, y_0, z_0)(z - z_0) = 0}$$

The <u>normal line</u> to S at P that is perpendicular to T_pS is given by:

$$\frac{x-x_0}{D_xF(P)} = \frac{y-y_0}{D_yF(P)} = \frac{z-z_0}{D_zF(P)} \qquad P=\langle x_0,y_0,z_0\rangle$$

<u>Ex</u>. <u>Special Case</u>: $F(x,y,z) = f(x,y) - z = 0$. This is the case when z is a function of x and y.

Then we obtain:

$$D_xF(P) = D_xF(x_0,y_0,z_0) = D_xf(x_0,y_0)$$

$$D_yF(x_0,y_0,z_0) = D_yf(x_0,y_0)$$

$$D_zF(x_0,y_0,z_0) = -1$$

So the equation of T_pS is

$$D_xf(x_0,y_0)(x-x_0) + D_yf(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

which is equivalent to our old formula for T_pS.

<u>Ex</u>. Find T_pS and the normal line to the ellipsoid

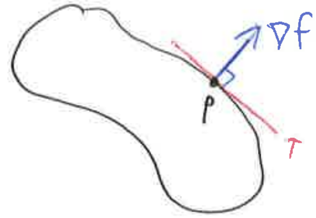
$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

at $(-2, 1, -3)$.

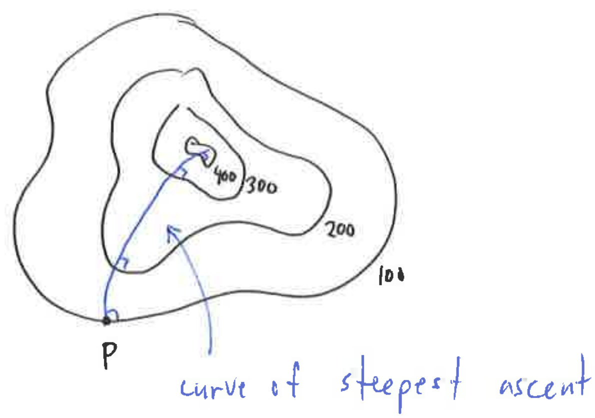
DO IT.

Based on all of this, we now see that the gradient vector has geometric significance as the direction of steepest ascent. If we have a contour map of a surface

One level curve:



Contour map:



The gradient vector is the tangent vector to the curve of steepest ascent at every point. This is the curve that runoff down a mountain will follow, for example.

Ex. (21) Find all points at which the fastest change of the function is (in the direction of) $\vec{i}+\vec{j}$. $f(x,y) = x^2+y^2-2x-4y$

$\nabla f = \langle 2x-2, 2y-4 \rangle$

$$\begin{cases} 2x-2 = 1 \\ 2y-4 = 1 \end{cases} \Rightarrow \quad \begin{array}{l} x = 3/2 \\ y = 5/2 \end{array}$$

$2x-2 = 2y-4$

$2x+2 = 2y$

$\boxed{y = x+1}$ All points on this line