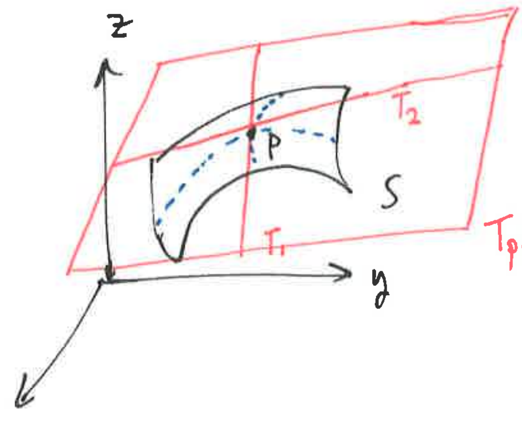
§11.4 Tangent Planes and linear approximations

Consider a surface S that is the graph of $z=f(x,y)$, and suppose f has continuous partial derivatives.

Let $P(x_0, y_0, z_0)$ be a point in $\Gamma(f) = S$.

The tangent lines T_1 to C_1 and T_2 to C_2 from section 11.3 determine a plane T_pS called the tangent plane to S at P.



T_pS = tangent plane to S at P.

Close to P, the surface S looks approximately like the plane T_pS, just like in Calc I a curve is locally approximated by the tangent line to a point.

The equation of T_pS is

$$z - z_0 = \partial_x f(x_0,y_0)(x-x_0) + \partial_y f(x_0,y_0)(y-y_0)$$

or

$$z = f(x_0,y_0) + \partial_x f(x_0,y_0)(x-x_0) + \partial_y f(x_0,y_0)(y-y_0)$$

* very similar to the eqn of a tan line in Calc I.

__ex.__ Find the tangent plane to $z = 2x^2 + y^2$ at the point $(1,1,3)$

$$f(x,y) = 2x^2 + y^2 \qquad z_0 = 3$$

$$\partial_x f(x,y) = 4x \qquad \partial_x f(x_0,y_0) = 4$$

$$\partial_y f(x,y) = 2y \qquad \partial_y f(x_0,y_0) = 2$$

So $$z = 3 + 4(x-1) + 2(y-1)$$

$$z = 3 + 4x - 4 + 2y - 2$$

$$\boxed{z = 4x + 2y - 3}$$

__Linear Approximations__

We can approximate values of $f(x,y) = z$ close to P by the tangent plane T_pS. For example, In the last problem

$$L(x,y) = 4x + 2y - 3$$

is the linearization of $f(x,y) = 2x^2 + y^2$ near (at) $(1,1,3)$.

Close to P $\quad f(x,y) \approx L(x,y)$.

For example $\quad f(1.1, 0.95) \approx L(1.1, 0.95) = 4(1.1) + 2(0.95) - 3$

$$= 4.4 + 1.9 - 3$$

$$= 3.3$$

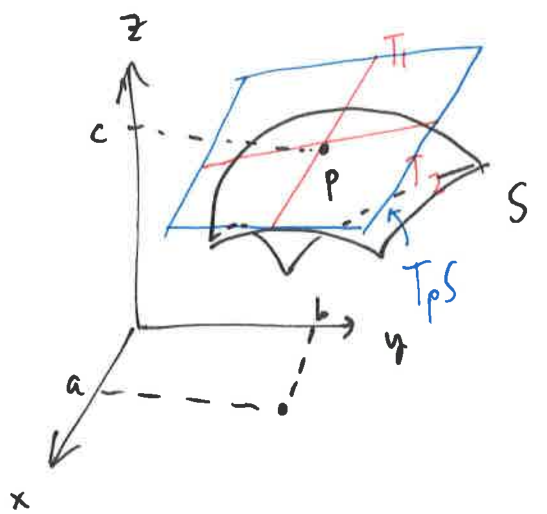
2/28

Plugging into f (and doing more work) we see that

$$f(1.1, 0.95) = 3.3225$$

Recall from last class:

T_pS = the tangent plane to S at the point p



$p = (a, b, c)$

$c = f(a, b)$

The tangent plane is given by the equation:

$$z = D_x f(a,b)(x-a) + D_y f(a,b)(y-b) + c$$

This is a real-valued function of x and y called the linearization of f ~~near~~ (at) the point (a,b).

We write $\boxed{L_p(x,y) = D_x f(a,b)(x-a) + D_y f(a,b)(y-b) + f(a,b)}$

or just $L(x,y)$

Near p, $f(x,y) \approx L(x,y)$.

Now, for a function of two variables

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

So Δz is the amount of change in the value of f when (x,y) is changed by $(\Delta x, \Delta y)$.

* Now write the def'n of differentiable.

This says that f is differentiable at a point if L is a "good" approximation at that point.

Theorem. If $D_x f$ and $D_y f$ exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Ex. Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$, and find L. Use it to approx. $f(1.1, -0.1)$.

$$D_x f = xye^{xy} + e^{xy} \qquad D_x f(1,0) = 0 + 1 = 1$$

$$D_y f = x^2 e^{xy} \qquad D_y f(1,0) = 1$$

Both $D_x f$ and $D_y f$ are continuous. Therefore f is differentiable.

The linearization of f is given by

$$L(x,y) = (x-1) + y + 1 = x + y$$

$L(1.1, -0.1) = 1.1 - 0.1$
$= 1$

$f(1.1, -0.1) \approx 0.98542$

Ex. 13 Find the linearization of $f(x,y) = \tan^{-1}(x+2y)$ at $(1,0)$

Need $f(1,0) = \arctan(1) = \pi/4$

$D_x f(1,0) = \dfrac{1}{1+1} = 1/2$

$D_y f(1,0) = \dfrac{2}{1+1} = 1$

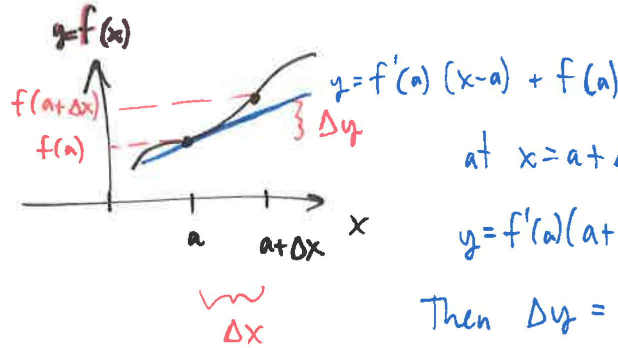
So $\boxed{L(x,y) = \frac{1}{2}(x-1) + y + \pi/4}$

Defn. If $z = f(x,y)$, then f is differentiable at (a,b) if Δz can be expressed in the form

$$\Delta z = D_x f(a,b)\Delta x + D_y f(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Recall from calc I:



for some small change in x, Δx, the change in y is

$$\Delta y = f(a+\Delta x) - f(a)$$

at $x = a + \Delta x$ we get

$y = f'(a)(a+\Delta x - a) + f(a)$

Then $\Delta y =$

$f'(a)\Delta x + f(a) - f(a)$

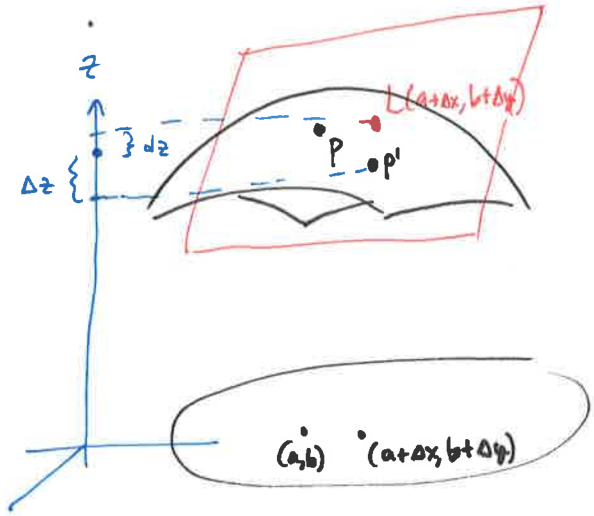
$= f'(a)\Delta x.$

In Chptr 2: If f is differentiable, then

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

In the limit we get the differential $dy = f'(a)\,dx$

Differentials:



Good picture on p. 622.

Let $dx = \Delta x$ and $dy = \Delta y$ be small changes (increments) in x and y. Then the differential (or total differential) is

$$df = dz = D_x f(x,y)\,dx + D_y f(x,y)\,dy$$
$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Suppose we let $dx = \Delta x = x-a$ and $dy = \Delta y = y-b$. Then dz is

$$dz = D_x f(a,b)(x-a) + D_y f(a,b)(y-b)$$

Thus in this notation,

$$f(x,y) \approx f(a,b) + dz \quad (\text{near } (a,b)!)$$

<u>Ex</u>. $z = f(x,y) = x^2 + 3xy - y^2$ Find dz.

$$D_x f = 2x + 3y$$
$$D_y f = 3x - 2y$$

$$dz = (2x + 3y)\,dx + (3x - 2y)\,dy$$

Find dz when $(x,y) = (2,3)$ and $(\Delta x, \Delta y) = (0.05, -0.04)$

$$dz = (2\cdot 2 + 3\cdot 3)(0.05) + \underset{0}{\cancel{(3\cdot 2 - 2\cdot 3)}}(-.04)$$
$$= 5.5(0.05)$$
$$= 0.65$$

Use this to approximate $f(2.05, 2.96)$

$$f(2.05, 2.96) \approx f(2,3) + dz$$
$$= 2^2 + 3(2)(3) - 3^2 + 0.65$$
$$= 4 + 18 - 9 + 0.65$$
$$= \underline{13.65}$$

<u>Ex</u>. The base radius and height of a right circular cone are measured to be 10 cm and 25 cm respectively, w/ error of up to 0.1 cm possible in each.
Use differentials to estimate the maximum area in the calculated volume.

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh = \frac{2}{3}\pi r h\,dr + \frac{1}{3}\pi r^2\,dh$$
$$= \frac{2}{3}\pi(10)(25)(0.1) + \frac{1}{3}\pi(10)^2(0.1) = \frac{1}{3}\pi(0.1)(500 + 100)$$
$$= \boxed{20\pi \text{ cm}^3}$$