§11.3: Partial Derivatives

f a function of two variables, x and y.

Suppose we fix $y=b$ and let x vary. Then we are really considering only a function of one variable: $g(x) = f(x,b)$.

If g has a derivative at $x=a$, we call this the partial derivative of f wrt x at (a,b).

We write

$$D_x f(a,b) = \partial_x f(a,b) = \frac{\partial}{\partial x} f(a,b) = f_x(a,b) = g'(a)$$

where $g(x) = f(x,b)$.

<u>Defn</u>. If f is a function of x and y, we define the partial derivatives of f by

$$\partial_x f(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and}$$

$$\partial_y f(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

* <u>Rule</u> for finding partial derivatives: to find $\partial_x f$, regard y as a constant and take the usual x-derivative.
Likewise for $\partial_y f$.

Ex. $f(x,y) = x^3 + x^2y^3 - 2y^2$.

Find $\partial_x f(2,1)$ and $\partial_y f(2,1)$.

$$\partial_x f(x,y) = 3x^2 + 2xy^3$$

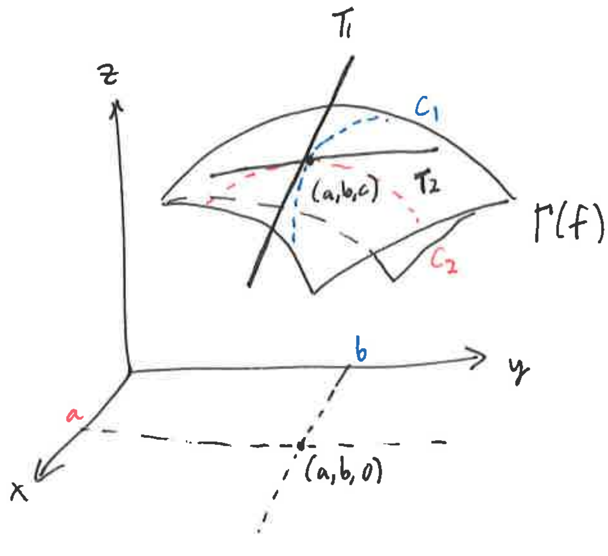
$$\partial_x f(2,1) = 3(2^2) + 2(2)(1^3)$$
$$= 12 + 4 = 16$$

$$\partial_y f(x,y) = 3x^2y^2 - 4y$$

$$\partial_y f(2,1) = 3(2)^2(1)^2 - 4(1)$$
$$= 12 - 4 = 8$$

Interpreting Partial Derivatives:

$\partial_x f(a,b)$ is the slope of the tangent line T_1, and $\partial_y f(a,b)$ is the slope of the tangent line T_2.



They tell you how much the surface is changing in those directions.

C_1 = trace of $\Gamma(f)$ along $y=b$

$c = f(a,b)$

C_2 = trace of $\Gamma(f)$ along $x = a$.

Ex. $f(x,y) = 4 - x^2 - 2y^2 = z$

Find $\partial_x f(1,1)$ and $\partial_y f(1,1)$, and interpret these as slopes.

$$\partial_x f(x,y) = -2x \qquad \partial_x f(1,1) = -2$$
$$\partial_y f(x,y) = -4y \qquad \partial_y f(1,1) = -4$$

when $y=1$, the trace is $4 - x^2 - 2 = z$

$z = -x^2 + 2$, $y = 1$.

Since $x=1$ also, the pt of intersection is $(1,1,1)$

the slope of the tangent line to this parabola is $\partial_x f(1,1) = -2$.

Similarly, when $x=1$, $z = 3 - 2y^2$.

the slope of the tangent line to this parabola at $(1,1,1)$ is $\partial_y f(1,1) = -4$.

Ex. $f(x,y) = \sin\left(\frac{x}{1+y}\right)$ Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \left(\frac{1}{1+y}\right)\cos\left(\frac{x}{1+y}\right)$$

$$\frac{\partial f}{\partial y} = -x(1+y)^{-2}\cos\left(\frac{x}{1+y}\right)$$

Ex. $x^3 + y^3 + z^3 + 6xyz = 1$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ implicitly.

$\frac{\partial z}{\partial x}$: $3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$

$$\frac{\partial z}{\partial x}(3z^2 + 6xy) = -3x^2 - 6yz$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}}$$

Similarly $\frac{\partial z}{\partial y}$.

Ex. Find $\partial_x f, \partial_y f, \partial_z f$ for $f(x,y,z) = e^{xy} \ln z$

$$\partial_x f = y e^{xy} \ln z$$

$$\partial_y f = x e^{xy} \ln z$$

$$\partial_z f = \frac{e^{xy}}{z}$$

Higher Derivatives:

$$\partial_x(\partial_y f) = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \quad \ldots \text{etc.}$$

$$\partial_x(\partial_x f) = f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$\partial_y(\partial_x f) = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

Ex. Find all 2^{nd} partials of

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

$$\partial_x f = 3x^2 + 2xy^3$$

$$\partial_y f = 3x^2y^2 - 4y$$

$$\left.\begin{aligned}
\partial_x(\partial_x f) &= 6x + 2y^3 \\
\partial_y(\partial_x f) &= 6xy^2 \\
\partial_y(\partial_y f) &= 6x^2y - 4 \\
\partial_x(\partial_y f) &= 6xy^2
\end{aligned}\right\}$$

Notice that

$$\partial_x(\partial_y f) = \partial_y(\partial_x f)$$

Clairaut's Thm. Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Ex. Calculate $\partial_z \partial_y \partial_x \partial_x \left[f(x,y,z) \right]$ for $f(x,y,z) = \sin(3x + yz)$.

FTIS.

PDE: Partial Differential Equations

PDE are equations involving partial derivatives of unknown functions.

One well-known PDE is <u>Laplace's Equation</u>:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solutions to Laplace's equation are called <u>harmonic functions</u>.

Related to physical phenomena like heat conduction, fluid flow, and electric potential.

<u>Ex</u> Show that $u(x,y) = e^x \sin y$ ~~satisfies~~ solves Laplace's equation.

Do it.

The <u>wave equation</u> is $\frac{\partial u^2}{\partial t^2} = a^2 \frac{\partial u^2}{\partial x^2}$

describes ocean waves, sound waves, light wave, etc.

<u>Ex</u>. Verify that $u(x,t) = \sin(x - at)$ ~~satisfies~~ solves. the wave eqn.

Do it.