# 10.7: Vector Functions and Space Curves

We've already seen the simplest example of this when we studied the eq'n of a line in space.

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

$\vec{r}$ is a function that assigns a vector to every input value for t.

In general, a <u>vector-valued function</u> looks like

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where f, g, h, are any functions (real-valued).

$\vec{r}$ is still a position vector for a particle moving along a curve, but now the curve could be much more complicated than a straight line.

<u>Ex</u>. $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

The <u>component functions</u> are

$$\begin{cases} x(t) = t^3 \\ y(t) = \ln(3-t) \\ z(t) = \sqrt{t} \end{cases}$$

The function $\vec{r}(t)$ is only defined when <u>all</u> components are.

$\text{dom}(x) = \mathbb{R}$, $\text{dom}(y) = (-\infty, 3)$, $\text{dom}(z) = [0, \infty)$,

therefore $\text{dom}(\vec{r}) = [0, 3)$ open!

The <u>limit</u> of a vector function $\vec{r}$ is <u>defined</u> by taking the limit of each component, provided that the limit of each component exists:

$$\lim_{t\to a} \vec{r}(t) := \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle .$$

---

<u>Ex</u>. $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \frac{\sin t}{t}\vec{k}$

Find $\displaystyle\lim_{t\to 0} \vec{r}(t)$

$$\lim_{t\to 0} \vec{r}(t) = \left[\lim_{t\to 0}(1+t^3)\right]\vec{i} + \left[\lim_{t\to 0} te^{-t}\right]\vec{j} + \left[\lim_{t\to 0}\frac{\sin t}{t}\right]\vec{k}$$

$$= 1\,\vec{i} + 0\,\vec{j} + 1\,\vec{k} = \vec{i} + \vec{k} = \langle 1, 0, 1\rangle .$$

---

A vector function $\vec{r}$ is <u>continuous</u> iff

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a) .$$

Notice then that $\vec{r}$ is continuous if and only if each of its component functions is.

The graph of a continuous vector function is called a <u>space curve</u>.

We can think of the graph $\gamma$ of $\vec{r}$ as a paramatrized curve (cf. Ch 9) with parameter $t$ and component functions as parametric equations.

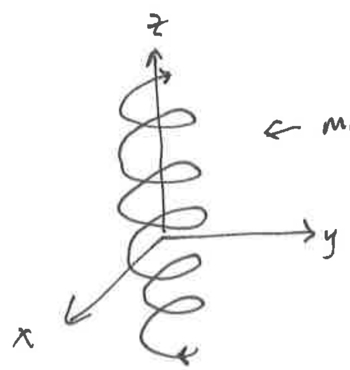
Ex. Describe the space curve: $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$.

Recall from § 10.5 that this is a straight line thru the point $(1, 2, -1)$ in the direction of $\langle 1, 5, 6 \rangle$.

Ex. Consider the space curve $\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\, \vec{k}$.

This is the polar parametrization of the circle in xy-plane: $\langle \cos t, \sin t \rangle$. Thus, a "top view" of this curve looks like a unit circle in xy-plane.

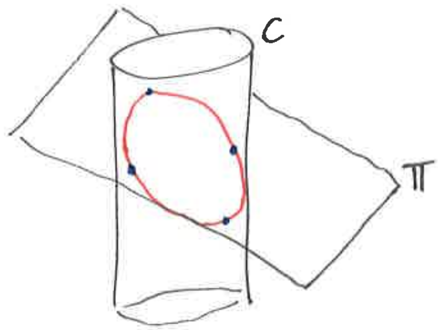
But every $t$ changes the z-height of the curve. The result is a helix:




← maybe not perfect drawing ...

Look at wolfram|Alpha.

Ex. Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2=1$ and plane $y+z=2$.




The projection of C onto xy-plane can be written as

$$\langle \cos t, \sin t\rangle \qquad 0 \le t \le 2\pi$$

To find a parametrization for z (in terms of t), write

$$y+z=2$$
$$z=2-y$$
$$z=2-\sin t$$

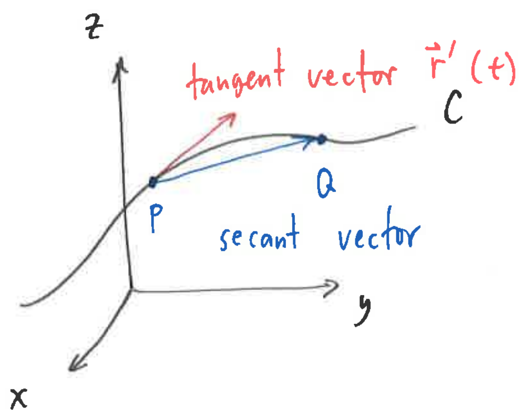
Then the corresponding vector function to this space curve is:

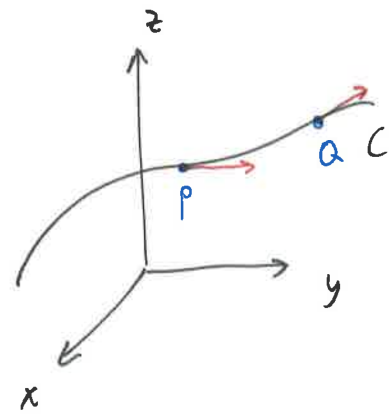
$$\vec{r}(t)=\langle \cos t, \sin t, 2-\sin t\rangle \qquad 0\le t\le 2\pi.$$

Now: Some Calculus!

The derivative $\boxed{\vec{r}\,'(t)=\dfrac{d\vec{r}}{dt}=\lim_{h\to 0}\dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}}$

It represents the limit of ~~the~~ secant lines (vectors) to a curve as the parameter approaches to t.




**Obvious Theorem.**

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ where $x, y, z$ are differentiable, then

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle.$$

**Proof.** Exercise. Use definition.

**Ex.** $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \sin 2t\, \vec{k}$

$$\vec{r}\,'(t) = \langle 3t^2, -te^{-t} + e^{-t}, 2\cos 2t \rangle$$

Find the unit tangent vector when $t = 0$.

$$\vec{r}\,'(0) = \langle 0, e^{-0}, 2\cos 0 \rangle = \langle 0, 1, 2 \rangle,$$

$$\vec{T}(0) = \frac{\vec{r}\,'(0)}{\|\vec{r}\,'(0)\|} = \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle.$$

$\vec{T}$ tells the direction that a particle is moving along the curve at that time.

$\underline{\text{Ex}}$. Find an equation for the tangent line to the helix

$$\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle$$

at the point $(0, 1, \pi/2)$

$$\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle$$
$$\vec{r}\,'(\pi/2) = \langle -2, 0, 1 \rangle$$

Then the tan line is given by:

$$\vec{r}_2(s) = \langle -2s, 1, \tfrac{\pi}{2} + s \rangle = \langle -2s, 1, \pi/2 + s \rangle$$

$\underline{\text{Thm}}$. $\underline{\text{Differentiation Rules}}$

Suppose $\vec{u}, \vec{v}$ are differentiable vector functions, $c \in \mathbb{R}$, and $f$ is a differentiable function.

1. $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \frac{d}{dt}\vec{u}(t) \pm \frac{d}{dt}\vec{v}(t)$

2. $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\frac{d}{dt}\vec{u}(t)$

3. $\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = \left[\frac{d}{dt}f(t)\right]\vec{u}(t) + f(t)\left[\frac{d}{dt}\vec{u}(t)\right]$

4. $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

5. $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

6. $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}\,'(f(t))$.

Ex. Show that if $\|\vec{r}(t)\| = c$ (constant), then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$ for all $t$.

Since $\vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = c^2$, and $c$ is constant, then taking the derivative we have

$$\frac{d}{dt}(c^2) = 0 = \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 2\,\vec{r}\,'(t)\cdot\vec{r}(t)$$

$\Rightarrow \vec{r}\,'\cdot\vec{r} = 0$ for all $t$

$\Rightarrow \vec{r}\,' \perp \vec{r}$ for all $t$.

---

<u>Definite Integrals</u>

If the component functions are integrable, then

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt,\ \int_a^b y(t)\,dt,\ \int_a^b z(t)\,dt \right\rangle$$

We can extend the <u>Fundamental Theorem of Calc</u>:

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big]_a^b = \vec{R}(b) - \vec{R}(a)$$

where $\vec{R}$ is an antiderivative of $\vec{r}$. (i.e., $\vec{R}\,' = \vec{r}$)

<u>Ex.</u> $\vec{r}(t) = \langle 2\cos t, \sin t, 2t \rangle$

$$\int \vec{r}(t)\,dt = \left\langle \int 2\cos t\,dt, \int \sin t\,dt, \int 2t\,dt \right\rangle$$

$$= \langle 2\sin t + c_1, -\cos t + c_2, t^2 + c_3 \rangle$$

$$= \langle 2\sin t, -\cos t, t^2 \rangle + \vec{c}$$

where $\vec{c} = \langle c_1, c_2, c_3 \rangle$ is a constant vector.

$$\int_0^{\pi/2} \vec{r}(t)\,dt = ?$$

$$= \langle 2\sin \pi/2, -\cos \pi/2, (\pi/2)^2 \rangle - \langle 2\sin 0, -\cos 0, 0 \rangle$$

$$= \langle 2-0, 0+1, \pi^2/4 \rangle$$

$$= \langle 2, 1, \pi^2/4 \rangle$$