Math 344: Calculus III
Chapter 12 Exam

Monday, 19 July 2013

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such.

e or

1. [5 points each] Determine whether the following statements are true or false. Simply write **T** or **F** in the blank.

a.) T $\displaystyle\int_2^5 \int_{10}^{20} x^2 \sin(x-y)\, dx\, dy = \int_{10}^{20} \int_2^5 x^2 \sin(x-y)\, dy\, dx$

b.) F $\displaystyle\int_0^1 \int_0^x \sqrt{x+y^2}\, dy\, dx = \int_0^x \int_0^1 \sqrt{x+y^2}\, dx\, dy.$

c.) F $\displaystyle\int_0^3 \int_{\sqrt{\pi}/2}^{\sqrt{\pi}} y e^{2x} \sin(y^2)\, dy\, dx = \int_0^2 e^{2x}\, dx \int_{\sqrt{\pi}/2}^{\sqrt{\pi}} y \sin(y^2)\, dy.$

d.) F The following integral represents the volume of the solid enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 3$.

$$\int_0^{2\pi} \int_0^3 \int_r^3 dz\, dr\, d\theta$$

2. [15 points] Evaluate the double integral.

$$\int_0^3 \int_{\sqrt{\pi}/2}^{\sqrt{\pi}} y e^{2x} \sin(y^2)\, dy\, dx$$

$$= \int_0^3 e^{2x}\, dx \int_{\frac{\sqrt{\pi}}{2}}^{\sqrt{\pi}} y \sin y^2\, dy$$

$$= \left(\tfrac{1}{2} e^{2x}\Big|_0^3\right)\left(-\tfrac{1}{2}\cos u \Big|_{\frac{\pi}{4}}^{\pi}\right) = -\tfrac{1}{4}\left[(e^6-1)\left(\cos\pi - \cos\tfrac{\pi}{4}\right)\right]$$

$$= -\tfrac{1}{4}\left[(e^6-1)\left(-1-\tfrac{\sqrt{2}}{2}\right)\right]$$

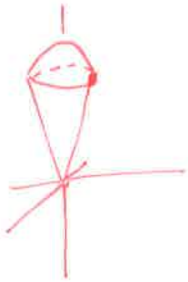
$$= -\tfrac{1}{4}\left[-e^6 + 1 - \tfrac{\sqrt{2}}{2}e^6 + \tfrac{\sqrt{2}}{2}\right]$$

$$\int_0^{\ln 3} \int_{\frac{\pi}{2}}^{\pi} e^x \sin y\, dy\, dx$$

$$= \int_0^{\ln 3} e^x\, dx \int_{\pi/2}^{\pi} \sin y\, dy$$

$$= \left(e^{\ln 3} - e^0\right)\left(-\cos\pi + \cos\tfrac{\pi}{2}\right) = (3-1)(1) = \boxed{2}$$

3. [15 points] Find the volume of the solid bounded below by the cone $z = \sqrt{x^2+y^2}$ and bounded above by the sphere $x^2+y^2+z^2=9$.



Spherical coords:

$\rho: 0 \to 3$

$\varphi: 0 \to \pi/4$

$\theta: 0 \to 2\pi$

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\rho^3 \sin\varphi \Big|_0^3 d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} 9 \sin\varphi \, d\varphi \, d\theta$$

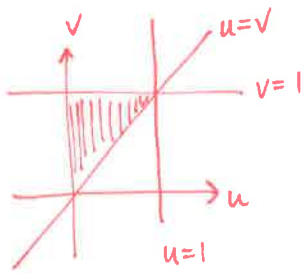
$$= \int_0^{2\pi} -9\cos\varphi \Big|_0^{\pi/4} d\theta = -9(2\pi)\left(\cos\tfrac{\pi}{4} - \cos 0\right)$$

$$= -18\pi\left(\frac{\sqrt{2}}{2} - 1\right)$$

$$= \boxed{18\pi - 9\pi\sqrt{2}}$$

4. [15 points] Evaluate the double integral.

$$\int_0^1 \int_u^1 \tfrac{1}{2} e^{v^2} \, dv \, du$$



$$= \int_0^1\int_0^v \frac{1}{2} e^{v^2} \, du \, dv = \frac{1}{2}\int_0^1 v e^{v^2} \, dv = \frac{1}{4} e^{u} \Big|_0^1$$

$$= \boxed{\frac{1}{4}(e-1)}$$

5. [15 points] Use spherical coordinates and a triple integral to find the volume between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.
[Hint: You can easily check your answer using high school geometry.]

$$\left.\begin{array}{l}\rho: 2 \to 3 \\ \varphi: 0 \to \pi \\ \theta: 0 \to 2\pi\end{array}\right\} \quad \text{Vol} = \int_0^{2\pi}\int_0^{\pi}\int_2^{3} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

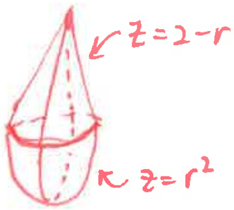
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} (27-8) \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{19}{3}\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi} \, d\theta$$

$$= \frac{38}{3}(2\pi) = \boxed{\frac{76\pi}{3}}$$

From geometry: $V = \frac{4}{3}\pi(R^3 - r^3) = \frac{4}{3}\pi(3^3 - 2^3) = \frac{4}{3}\pi(19) = \frac{76\pi}{3}$ ✓

6. [15 points] Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the cone $z = 2 - \sqrt{x^2 + y^2}$.



$$\left.\begin{array}{l} r: 0 \to 1 \\ \theta: 0 \to 2\pi \end{array}\right\} \quad V = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r} r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2-r-r^2) r \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r - r^2 - r^3 \, dr\, d\theta$$

$$= \int_0^{2\pi} r^2 - \frac{1}{3}r^3 - \frac{1}{4}r^4 \Big|_0^1 \, d\theta$$

$$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{10\pi}{12} = \boxed{\frac{5\pi}{6}}$$

7. [15 points] Find the area of the region in the plane bounded by the carioid

$$r = 1 - sin\theta$$

$$A = \int_0^{2\pi}\int_0^{1-\sin\theta} r \, dr\, d\theta = \frac{1}{2}\int_0^{2\pi} (1-\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 - 2\sin\theta + \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 - 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta) \, d\theta$$

$$= \frac{1}{2}\left[\theta + 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]\Big|_0^{2\pi}$$

$$= \text{[illegible]} = \frac{1}{2}[3\pi] = \frac{3\pi}{2}$$