Math 344: Calculus III
Chapter 11 Exam

Due: Monday, 8 July 2013

Late exams will **NOT** be accepted.

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such.

1. Find all first partial derivatives.

a.) $f(x,y) = x^2 \ln(x^2+y^2)$

$$\partial_x f = 2x \ln(x^2+y^2) + \frac{2x^3}{x^2+y^2}$$

$$\partial_y f = \frac{2x^2 y}{x^2+y^2}$$

b.) $g(u,v) = \dfrac{u+2v}{u^2+v^2}$

$$\partial_u g = \frac{(u^2+v^2) - 2u(u+2v)}{(u^2+v^2)^2} = \frac{-u^2+v^2-4uv}{(u^2+v^2)^2}$$

$$\partial_v g = \frac{2(u^2+v^2) - 2v(u+2v)}{(u^2+v^2)^2} = \frac{2u^2-2uv-2v^2}{(u^2+v^2)^2}$$

2. Find an equation of the tangent plane to the surface $z = e^x \cos y$ at the point $(0, 0, 1)$.

T_pS: $z - z_0 = \partial_x z(P)(x - x_0) + \partial_y z(P)(y - y_0)$

$\partial_x z = e^x \cos y \Rightarrow \partial_x z(p) = e^0 \cos(0) = 1$

$\partial_y z = -e^x \sin y \Rightarrow \partial_y z(p) = -e^0 \sin(0) = 0$

Thus, T_pS is given by

$$z - 1 = 1(x - 0) + 0(y - 0)$$

or $\boxed{z = 1 + x}$

3. Find du if $u = \ln(1 + s\,e^{2t})$.

$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial s} ds$

So, $\boxed{du = \frac{2se^{2t}}{1+se^{2t}} dt + \frac{e^{2t}}{1+se^{2t}} ds}$

4. Find the linearization of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 2, 6)$ and use it to approximate $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

$f(P) = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

$\partial_x f = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}}, \quad \partial_y f = \dfrac{2y}{2\sqrt{x^2+y^2+z^2}}, \quad \partial_z f = \dfrac{2z}{2\sqrt{x^2+y^2+z^2}}$

$\partial_x f(P) = \frac{3}{7} \qquad \partial_y f(P) = \frac{2}{7} \qquad \partial_z f(P) = \frac{6}{7}$

$$\boxed{L(x,y,z) = 7 + \tfrac{3}{7}(x-3) + \tfrac{2}{7}(y-2) + \tfrac{6}{7}(z-6)}$$

$$L(3.02, 1.97, 5.99) = 7 + \frac{3}{7}\left(\frac{2}{100}\right) + \frac{2}{7}\left(\frac{-3}{100}\right) + \frac{6}{7}\left(\frac{-1}{100}\right)$$

$$= 7 + \frac{6-6-6}{700} = 7 - \frac{6}{700} = \frac{4900-6}{700} = \boxed{\frac{4894}{700}}$$

5. Show that the limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^6}{x^4 + y^{12}}$$

<u>Path 1:</u> $x = 0$: $\displaystyle\lim_{y\to 0} \frac{0}{y^{12}} = 0$.

<u>Path 2:</u> $x = y^3$: $\displaystyle\lim_{y\to 0} \frac{y^{12}}{y^{12}+y^{12}} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$

$0 \neq 1/2$, so the lim DNE.

6. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2,1,3)$. You don't have an equation for S, but you know that the curves

$$\begin{aligned} r_1(t) &= \langle 2+3t,\ 1-t^2,\ 3-4t+t^2\rangle, \text{ and} \qquad \vec{r}_1(0)=P \\ r_2(u) &= \langle 1+u^2,\ 2u^3-1,\ 2u+1\rangle \qquad \vec{r}_2(1)=P \end{aligned}$$

both lie on S and pass through P. Find an equation of the tangent plane T_PS.

$\dot{\vec{r}}_1 = \langle 3, -2t, -4+2t\rangle$, $\dot{\vec{r}}_1(P) = \langle 3, 0, -4\rangle$

$\dot{\vec{r}}_2 = \langle 2u, 6u^2, 2\rangle$, $\dot{\vec{r}}_2(P) = \langle 2, 6, 2\rangle$

$\vec{n} = \dot{\vec{r}}_1 \times \dot{\vec{r}}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = -24\vec{i} - 14\vec{j} + 18\vec{k}$

$\vec{n} = \langle -24, -14, 18\rangle$ or $\langle -12, -7, 9\rangle$

T_PS: $\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$: $\langle -12,-7,9\rangle \cdot \langle x-2, y-1, z-3\rangle = 0$

$\Rightarrow -12x + 24 - 7y + 7 + 9z - 27 = 0$

or $\boxed{12x + 7y - 9z = 4}$

7. Find the directional derivative of $f(x,y) = x^2e^{-y}$ at the point $(-2,0)$, in the direction of the point $(2,-3)$.

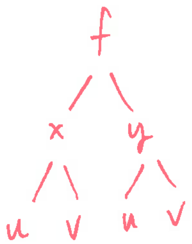
$\nabla f(x,y) = \langle 2xe^{-y}, -x^2e^{-y}\rangle$

$\nabla f(P) = \langle -4, -4\rangle$

$\vec{v} = \langle 4, -3\rangle \Rightarrow \vec{u} = \langle \frac{4}{5}, \frac{-3}{5}\rangle$

$D_{\vec{u}}f(P) = \nabla f(P)\cdot\vec{u} = \langle -4,-4\rangle\cdot\langle \frac{4}{5}, \frac{-3}{5}\rangle = \frac{-16}{5} + \frac{12}{5} = \boxed{\frac{-4}{5}}$

8. Let $f = f(x, y)$, $x = x(u, v)$, and $y = y(u, v)$. Make a "chain rule tree" for f and $\partial f/\partial u$, then write down a symbolic formula for $\partial^2 f/\partial u^2$.

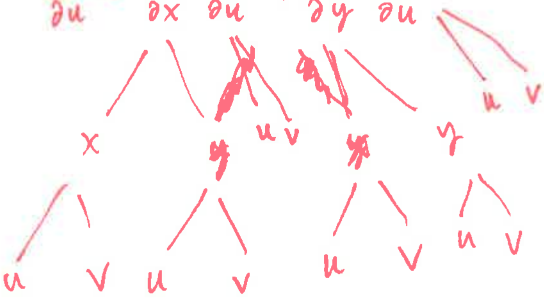


$$\frac{\partial^2 f}{\partial u^2} = \cancel{\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial u}\right]\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial u}\right]\frac{\partial y}{\partial u}}$$

$$\cancel{= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\right]\frac{\partial x}{\partial u} + \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right]\frac{\partial x}{\partial u} + \ldots}$$

The Chain Rule Strikes again! Ugh ^^

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$



... sorry ^^

$$\frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\right] + \frac{\partial}{\partial u}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right]$$

$$= \left[\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial u}\right]\frac{\partial x}{\partial u} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial u^2} + \left[\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u}\right]\frac{\partial y}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial u^2}$$

$$= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial u}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2 + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial u^2}$$

9. Prove the theorem: *Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\boldsymbol{u}}f(\boldsymbol{x})$ is $\|\nabla f(\boldsymbol{x})\|$, and it occurs when $\boldsymbol{u}$ has the same direction as the gradient vector $f(\boldsymbol{x})$.*

$$D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} = \|\nabla f(\vec{x})\| \, \cancel{\|\vec{u}\|}^{\,1} \cos\theta = \|\nabla f(\vec{x})\| \cos\theta$$

This is maximum when $\theta = 0$ and $\cos\theta = 1$, whence

$$\max\{D_{\vec{u}} f(\vec{x})\} = \|\nabla f(\vec{x})\| \quad \text{and} \quad \nabla f(\vec{x}) \parallel \vec{u}. \blacksquare$$

10. *a.*) Find the gradient of f; *b.*) Evaluate the gradient at the point P; *c.*) Find the rate of change of f at the point P in the direction of the vector $\boldsymbol{u}$.

$$\begin{aligned} f(x,y) &= \sin(2x+3y) \\ P &= (-6,4) \\ u &= (\sqrt{3}i - j) \end{aligned}$$

a) $\nabla f(x,y) = \langle 2\cos(2x+3y), \ 3\cos(2x+3y) \rangle$

b) $\nabla f(P) = \langle 2\cos(0), \ 3\cos(0) \rangle = \langle 2, 3 \rangle$

c) $\|\vec{u}\| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$. Use $\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \rangle$ instead

$D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = \langle 2, 3 \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \rangle = \boxed{\sqrt{3} - \frac{3}{2}}$

11. Find the maximum rate of change of the function $f(s,t) = t\,e^{st}$ at the point $(0,2)$, and give the direction that it occurs.

$\nabla f = \langle t^2 e^{st}, \ e^{st} + tse^{st} \rangle$

$\nabla f(0,2) = \langle 4, 1 \rangle$

$\boxed{\|\nabla f(0,2)\| = \sqrt{17}} = \max \{ D_{\vec{u}} f(P) \}$

The direction is $\boxed{\vec{u} = \langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle}$