Math 344: Calculus III
Chapter 10 Exam

Friday, 21 June 2013

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded based not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such.

1. [10 points] Consider the vectors $\boldsymbol{u} = \langle 3, -1, 7\rangle$ and $\boldsymbol{v} = \langle -5, 0, 2\rangle$.

a.) Find $\boldsymbol{u} \cdot \boldsymbol{v}$

$$\vec{u} \cdot \vec{v} = \langle 3,-1,7\rangle \cdot \langle -5, 0, 2\rangle = -15 + 0 + 14 = \boxed{-1}$$

b.) Find $\boldsymbol{u} \times \boldsymbol{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 7 \\ -5 & 0 & 2 \end{vmatrix} = \vec{i}(-2) - \vec{j}(41) + \vec{k}(-5)$$

$$= \boxed{\langle -2, -41, -5\rangle}$$

c.) Find $\text{proj}_{\boldsymbol{v}} \boldsymbol{u}$ $= \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \dfrac{-1}{5^2 + 2^2} \langle -5, 0, 2\rangle = \boxed{\left\langle \dfrac{5}{29}, 0, \dfrac{-2}{29} \right\rangle}$

2. [10 points] *a.)* Find the equation of the plane Π that passes through the points $P(0,-1,1)$, $Q(1,0,2)$, and $R(-2,-2,2)$.

$\vec{PQ} = \langle 1,1,1 \rangle$

$\vec{PR} = \langle -2,-1,1 \rangle$

$\vec{QR} = \langle -3,-2,0 \rangle$

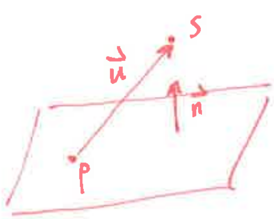
$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = \vec{i}(2) - \vec{j}(3) + \vec{k}(1) = \langle 2,-3,1 \rangle$$

Thus, $\Pi: \ 2x - 3y + z - \big(2(0) + (-3)(-1) + (1)(1)\big) = 0$

or

$$\boxed{2x - 3y + z = 4}$$

b.) Find the distance between the plane Π from part *a.)* and the point $(1,1,1)$.



$$D = \| \text{proj}_{\vec{n}} \vec{u} \| = \frac{|\vec{u} \cdot \vec{n}|}{\|\vec{n}\|}$$

$\vec{u} = \langle 1-0, 1+1, 1-1 \rangle = \langle 1,2,0 \rangle$

$\vec{u} \cdot \vec{n} = \langle 1,2,0 \rangle \cdot \langle 2,-3,1 \rangle = 2-6 = -4$

$|\vec{u} \cdot \vec{n}| = 4$

$\|\vec{n}\| = \sqrt{2^2+3^2+1} = \sqrt{14}$

so $\boxed{D = \dfrac{4}{\sqrt{14}}}$

3. [20 points] Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{R}^3$. Prove the Cauchy-Schwarz Inequality:

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|.$$

When is it an actual equality?

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta,$$

so $$|\vec{u} \cdot \vec{v}| = \big|\|\vec{u}\| \, \|\vec{v}\| \cos\theta\big| = \|\vec{u}\| \, \|\vec{v}\| \, |\cos\theta|$$

$$\leq \|\vec{u}\| \, \|\vec{v}\| \quad \text{since } |\cos\theta| \leq 1. \quad \blacksquare$$

$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \, \|\vec{v}\|$ if and only if $\vec{u} \parallel \vec{v}$.

4. [20 points] Consider the vector function $\boldsymbol{r}(t) = \langle \cos t, 3t, 2\sin(2t) \rangle$. Find $\dot{\boldsymbol{r}}(t)$, then find the unit tangent vector $\mathbf{T}(0)$.

$$\boxed{\dot{\vec{r}}(t) = \langle -\sin t, 3, 4\cos(2t) \rangle}$$

$$\dot{\vec{r}}(0) = \langle 0, 3, 4 \rangle$$

$$\vec{T}(0) = \frac{\dot{\vec{r}}(0)}{\|\dot{\vec{r}}(0)\|} = \frac{\langle 0,3,4 \rangle}{\sqrt{3^2+4^2}} = \boxed{\left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle}$$

5. [20 points] Let $f(t) = \boldsymbol{u}(t) \cdot \boldsymbol{v}(t)$, where $\boldsymbol{u}(t) = \langle 3t+1,\, 2,\, 4t-1 \rangle$ and $\boldsymbol{v}(t) = \langle t,\, t^2,\, t^3 \rangle$. Find $f'(2)$.

$$f(t) = \langle 3t+1, 2, 4t-1 \rangle \cdot \langle t, t^2, t^3 \rangle = 3t^2 + t + 2t^2 + 4t^4 - t^3$$

$$= 4t^4 - t^3 + 5t^2 + t$$

$$f'(t) = 16t^3 - 3t^2 + 10t + 1$$

$$f'(2) = 16(8) - 3(4) + 10(2) + 1$$

$$= 128 - 12 + 20 + 1 = \boxed{137}$$

6. [10 points] Reduce the equation to one of the standard forms and classify the surface.

$$x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$$

Sketch the surface and some traces for 2 extra points.

$$x^2 - 2x \underline{+1} \quad -y^2 + 2y \underline{-1} \quad + z^2 + 4z \underline{+4} = -2 + 1 - 1 + 4$$

$$(x-1)^2 - (y-1)^2 + (z+2)^2 = 2$$

$$\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2} + \frac{(z+2)^2}{2} = 1$$

$\Rightarrow$ hyperboloid of one sheet

7. [20 points] Consider the helix parametrized by

$$r(t) = \langle 4t,\ 3\sin t,\ 3\cos t\rangle\,, \quad t > 0.$$

Find the arc length function $s(t) = L(r \mid [0,t])$, reparametrize r with respect to s, then find the curvature function $\kappa(s)$.

$$s(t) = \int_0^t \|\dot{\vec{r}}(u)\|\,du = \int_0^t \sqrt{16 + 9\cos^2 u + 9\sin^2 u}\,du$$

$$= \int_0^t \sqrt{16+9}\,du = 5t$$

so $\boxed{s = 5t} \Rightarrow t = \frac{s}{5}$ and

$$\boxed{\vec{r}(s) = \left\langle \frac{4}{5}s,\ 3\sin\left(\frac{s}{5}\right),\ 3\cos\left(\frac{s}{5}\right)\right\rangle}$$

$$\kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\|$$

$$\vec{T} = \frac{d\vec{r}}{ds} = \left\langle \frac{4}{5},\ \frac{3}{5}\cos\left(\frac{s}{5}\right),\ -\frac{3}{5}\sin\left(\frac{s}{5}\right)\right\rangle$$

$$\frac{d\vec{T}}{ds} = \left\langle 0,\ \frac{-3}{25}\sin\left(\frac{s}{5}\right),\ \frac{-3}{25}\cos\left(\frac{s}{5}\right)\right\rangle$$

and $\boxed{\kappa(s) = \left\|\frac{d\vec{T}}{ds}\right\| = \frac{3}{25}}$