Calculus III Project: Div, Grad, Curl, and all that...

Due date: Tuesday, 14 May 13, 8.00 am

This project closely follows section 13.5 in our book. You may find it useful to have the book open to this section as you work your way through these problems.

This is a "double project" worth 4% toward your overall grade in this course. Complete all of the items marked as **Ex** in the space provided on this paper.

Please complete all exercises neatly on this paper.

For the entirety of these notes, we will assume that we are working in an open, connected domain D in $\mathbb{R}^3$. Sometimes we will want all of $\mathbb{R}^3$ to be our domain, in which case we simply write $\mathbb{R}^3$ instead of D.

gradient and the del operator

Let $f : D \to \mathbb{R}$ be a smooth function. Recall that the *gradient* is an operator that sends the function f to the vector field

$$\operatorname{grad} f = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

Think of ∇ as an operator that takes in a smooth function on D and returns a vector field on D. The *del operator* ∇ is given by

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

This definition extends to $\mathbb{R}^n$ for any n in the obvious way. The del operator is the standard *differential operator* that we will use to construct a number of other operators.

curl

In $\mathbb{R}^3$ we can define something called the curl of a vector field. Let $\mathbf{F}(x,y,z) = P(x,y,z)\boldsymbol{i} + Q(x,y,z)\boldsymbol{j} + R(x,y,z)\boldsymbol{k}$. We define the *curl* of $\mathbf{F}$ to be the vector field on D defined by

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F},$$

where $\times$ is the usual cross product in $\mathbb{R}^3$. In matrix form this equation is given by

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

Ex 1 Find $\operatorname{curl}\mathbf{F}$ for $\mathbf{F} = \langle 1,\, x+y,\, xy-\sqrt{z}\,\rangle$.

Theorem 2 *If* $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ *has continuous second-order partial derivatives, then* $\operatorname{curl}(\nabla f) = \mathbf{0}$.

That is, if $\mathbf{F}$ is conservative, then $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

Ex 3 **Proof:**

The next theorem is the converse of the previous one. As with two-dimensional vector fields, this direction requires some more assumptions.

Theorem 4 *If* $\mathbf{F}$ *is a vector field defined on a simply connected domain* $D \subseteq \mathbb{R}^3$ *whose components have continuous first-order partial derivatives and* $\operatorname{curl} \mathbf{F} = \mathbf{0}$, *then* $\mathbf{F}$ *is conservative.*

I won't make you prove this one.

Ex 5 Verify that $\mathbf{F} = \left\langle 2xy,\, (x^2 + 2yz),\, y^2 \right\rangle$ is conservative. Find its potential function.

divergence

Let $\mathbf{F} = \langle P, Q, R \rangle$ on $D \subseteq \mathbb{R}^3$ and suppose $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ exist. The *divergence* of $\mathbf{F}$ is the function on D given by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Ex 6 Find the div $\mathbf{F}$ for $\mathbf{F} = \langle 1,\, x + y,\, xy - \sqrt{z} \rangle$.

The next theorem says that, in $\mathbb{R}^3$, vector fields that are the curl of another vector field, lie in the kernel of the divergence operator (whatever that means...).

Theorem 7 *If* $\mathbf{F} = \langle P, Q, R \rangle$ *is a vector field on* $\mathbb{R}^3$ *and* P, Q, R *have continuous second-order partial derivatives, then*

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0.$$

Ex 8 **Proof:**

If $\mathbf{F}(x, y, z)$ represents the flow of a fluid, then $\operatorname{div} \mathbf{F}(x, y, z)$ measures the tendency of the fluid to "diverge" from the point (x, y, z); hence the name. If $\operatorname{div} \mathbf{F} = 0$, then the $\mathbf{F}$ is called *incompressible.*

the Laplacian

Another important differential operator is called the Laplacian. This takes in smooth functions (at least twice differentiable) as arguments, and outputs another function. It is given by

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \operatorname{div}(\nabla f) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Some physicists and applied mathematicians write the Laplacian as ∇^2 instead of Δ since it is the dot product of two del operators. It also emphasizes the fact the the Laplacian is the sum of the second derivatives of f. That said, I personally don't think it's a very good idea to write it this way.

Ex 9 A function $f : D \to \mathbb{R}$ that satisfies $\Delta f = 0$ is said to be *harmonic* on D. Verify that $f(x, y) = e^x \sin y$ is harmonic on $\mathbb{R}$.

Green's Theorem revisited

There are two ways to rewrite Green's Theorem in the language of the differential operators that we have studied in these notes. First, let $\boldsymbol{k} = \langle 0, 0, 1\rangle$ be the unit vector in the z-direction, and $\mathbf{F}$ a vector field on $D \subseteq \mathbb{R}^2$. Then Green's Theorem can be rewritten as

$$\int_{\partial D} \mathbf{F} \cdot d\boldsymbol{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \boldsymbol{k} \, dA.$$

To make sense of this, we need to consider the 2-vector field $\mathbf{F}$ as a 3-vector field by making it constant (zero) in the z slot.

Ex 10 Proof:

There is another way to rewrite Green's Theorem. For this version we need to introduce the *normal vector field* to D along its boundary ∂D. This vector field is given by

$$\boldsymbol{n}(t) = \frac{\langle \dot{y}(t), -\dot{x}(t)\rangle}{\|\dot{\boldsymbol{r}}(t)\|}$$

where $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$ is a parametrization of ∂D. In this situation, Green's Theorem can be rewritten as

$$\int_{\partial D} \mathbf{F} \cdot \boldsymbol{n} \, ds = \iint_D \operatorname{div} \mathbf{F} \, dA.$$

Ex 11 Proof:

Ex 12 Using the last version of Green's Theorem, prove *Green's first identity*:

$$\iint_D f \Delta g \, dA = \int_{\partial D} f(\nabla g) \cdot \boldsymbol{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA,$$

where D is as in Green's Theorem, and the appropriate partial derivatives of f and g exist and are continuous. Notice: This looks like an Integration-by-Parts-like formula for area integrals! (Especially if you replace the Δ by ∇^2.)

Ex 13 Use Green's first identity to prove *Green's second identity*:

$$\iint_D (f \Delta g - g \Delta f) \, dA = \int_{\partial D} (f \nabla g - g \nabla f) \cdot \boldsymbol{n} \, ds$$

where all of the same assumptions on D, f, and g are satisfied.