Calculus III: Project 2

Due date: Tues, 19 February 13

Instructions: On your own paper, solve the following problems in a clean, neat, clear, organized, legible, *etc* manner. Your assignment will be graded on the presentation and validity of your work.

1. Show that the curvature κ is related to the tangent and normal vectors by the equation

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}.$$

2. (a) Show that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{B}$. (b) Show that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{T}$. (c) Deduce from parts (a) and (b) that $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$ for some number (function of s) $\tau(s)$ called the *torsion* of the curve. (d) Show that for a plane curve (one that can be reparametrized to lie in $\mathbb{R}^2$), $\tau(s) = 0$.

3. The following formulas, called the *Frenet-Serret formulas*, are of fundamental importance in differential geometry:

1. $d\mathbf{T}/ds = \kappa\mathbf{N}$
2. $d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B}$
3. $d\mathbf{B}/ds = -\tau\mathbf{N}$

Formula 1 comes from problem 1, and formula 3 comes from problem 2. Use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ to deduce formula 2 from formulas 1 and 3.

4. The Frenet-Serret formulas imply the following formula for torsion:

$$\tau = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \dddot{\mathbf{r}}}{\|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\|^2}.$$

(You can just believe this.) Use this formula to show that the circular helix

$$\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$$

where a and b are positive constants, has constant curvature and constant torsion.