## 13.4: Green's Theorem

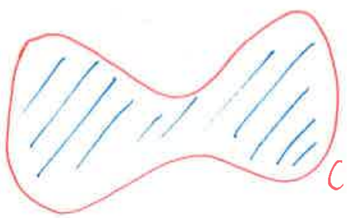
In this section we will study the relationship between path integrals around a simple closed path $C$ and double integrals over the simply connected domain $D$ that $C$ encloses. We'll start by stating the (in)famous *Jordan Curve Theorem.*

**Theorem 19** *Every simple closed curve in* $\mathbb{R}^2$ *divides the plane into exactly two connected regions, one that is bounded* (the interior), *and one that is not bounded* (the exterior).

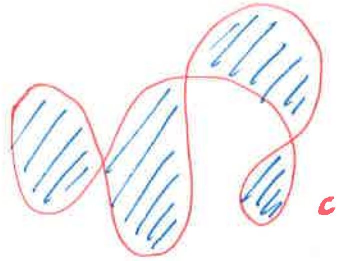
This theorem is "infamous" because, although it seems obvious, it turned out to be extremely hard to prove. In fact the original proof that was presented by Camille Jordan (whom the theorem is named after) turned out to be flawed. Eventually, after many others had tried and failed, Oswald Veblen was able to finally give a rigorous proof. Even today, some mathematicians disagree over who should get credit for the first correct proof. Don't worry, we won't try to prove it here. In honor of this theorem, we frequently call simple closed curves *Jordan curves.*

Let $C$ be a Jordan curve that encloses a domain $D$. By convention, we will say that $C$ has *positive orientation*, or $C$ is *positively oriented*, if we traverse $C$ in the counter-clockwise direction. Therefore, the domain $D$ is always on the left hand side, or driver's side (in the US, at least), of the path.

Draw some pictures here:

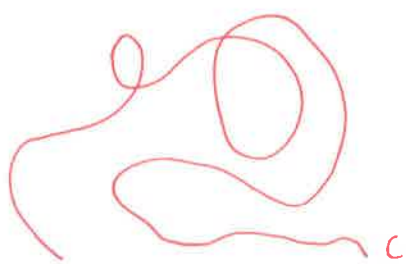



simple closed

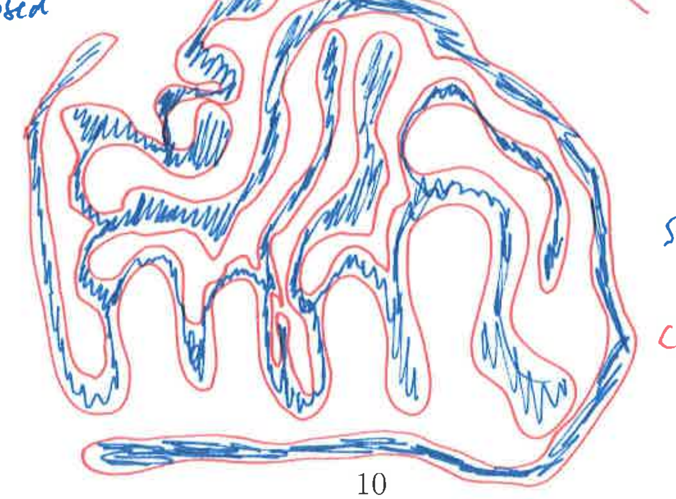



not simple, closed


simple, not closed




not simple, not closed




Supposed to be: simple closed
I may have crossed a couple times because I can't draw

**Theorem 20** (Green's Theorem.) *Let $C$ be a positively oriented, piecewise smooth Jordan curve in the plane, and $D$ the domain enclosed by it. If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ has continuous partial derivatives on an open region containing $D$, then*

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{11}$$

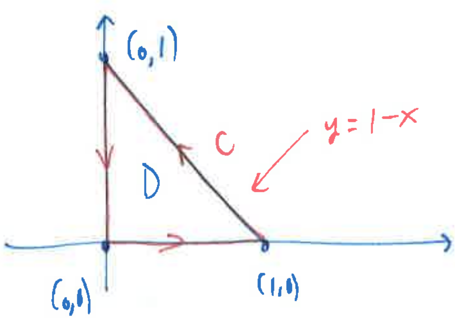
A notation frequently used in physics and applied math to denote the boundary of a domain $D$ is $\partial D$. In this case $\partial D = C$, so we could rewrite the equation in the theorem as

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{12}$$

to emphasize the relationship that this theorem provides between a domain and its boundary.

**RE 21** Read through the proof on p. 752f carefully, then do exercise 28, p. 757.

**Ex 22** Evaluate $\int_C x^4\,dx + xy\,dy$, where $C$ is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$.




$\dfrac{\partial Q}{\partial x} = y \qquad \dfrac{\partial P}{\partial y} = 0$

So:

$$\int_C x^4 dx + xy\,dy = \iint_D y\,dA$$

$$= \int_0^1 \int_0^{1-x} y\,dy\,dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 1 - 2x + x^2\,dx$$

$$= \frac{1}{2}\left(x - x^2 + \frac{1}{3}x^2\right) = \boxed{\frac{1}{6}}$$

**Ex 23** Let $\mathbf{F} = (3y - e^{\sin x})\,\boldsymbol{i} + (7x + \sqrt{y^4+1})\,\boldsymbol{j}$. Evaluate $\int_C \mathbf{F}\cdot d\boldsymbol{r}$, where $C$ is the circle $x^2+y^2=9$.

circle w/ radius 3.

$$\frac{\partial Q}{\partial x} = 7 \quad \frac{\partial P}{\partial y} = 3 \quad \Rightarrow \int_C \vec{F}\cdot d\vec{r} = \iint_D 4\, dA = 4\iint_D dA = 4(\pi r^2)$$

$$= 4(\pi \cdot 3^2)$$

$$= \boxed{36\pi}$$

That was easy! What if you tried to do the path integral directly?

Both of these examples use Green's Theorem to convert a path integral to a double integral, but sometimes it's useful to use Green's Theorem in the other direction.

**Ex 24** The area of a domain $D$ is given by $\iint_D dA$. Use Green's Theorem to find at least 3 different path integral formulas for the area of $D$.

We need a $P$ and $Q$ s.t. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

Some choices:

$P = 0 \quad Q = x$

$P = -y \quad Q = 0$

$P = -\frac{1}{2}y \quad Q = \frac{1}{2}x$

$$\text{Then } A(D) = \iint_D dA = \int_{\partial D} x\,dy = -\iint_{\partial D} y\,dx = \iint_{\partial D} -\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy$$

**Ex 25** Find the area enclosed by the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Let's use $A(D) = \iint_{\partial D} x\,dy$.

In parametric eqn's the ellipse is given by $\left.\begin{array}{l} x = a\cos t \\ y = b\sin t \end{array}\right\}\ 0 \le t \le 2\pi$

$\dot{y} = b\cos t$

$$= \int_a^b x(t)\,\dot{y}(t)\,dt$$

$$= \int_0^{2\pi} a\cos t \cdot b\cos t\,dt$$

$$= ab\int_0^{2\pi} \cos^2 t\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} 1 + \cos 2t\,dt = \frac{ab}{\cancel{2}}\left(\cancel{2}\pi + \cancel{\tfrac{1}{2}\sin 2\pi} - \cancel{\tfrac{1}{2}\sin 0}\right) = \boxed{\pi ab}$$

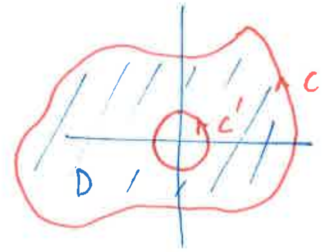
Yup!

**Ex 26** Let

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

Show that $\int_C \mathbf{F} \cdot dr = 2\pi$ for every positively oriented Jordan curve that encloses the origin.

$$\frac{\partial Q}{\partial x} = \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2} \qquad \frac{\partial P}{\partial y} = \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2}$$




$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-x^2+y^2 \quad +x^2 - y^2}{(x^2+y^2)^2} = 0$$

$\partial D = C \cup (-C')$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} + \int_{-C'} \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D 0 \, dA = 0.$$

Therefore $\int_C \vec{F} \cdot d\vec{r} = -\int_{-C'} \vec{F} \cdot d\vec{r}$

$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}$

For any curve $C$ and "small enough" circle $C'$.

Now, Calculate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{r}(t) = \langle a\cos t, a \sin t \rangle$
$0 \le a \le 2\pi$.

**RE 27** Use Green's Theorem to prove Theorem 13 when $C$ is a Jordan curve. Can you think of a way to prove it for *any* closed curve?

Hint: look at our picture of a closed, not simple curve on p.10 of the notes.