13.3: The Fundamental Theorem for Path Integrals

One half of the Fundamental Theorem of Calculus from Calculus 1 is the Evaluation Theorem. It says that if $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx} F(x)\,dx = F(b) - F(a), \tag{9}$$

where F is any antiderivative of f; *i.e.*, $f(x) = F'(x)$ for all $x \in [a, b]$. We expect a similar theorem for path integrals.

Let $\mathbf{F}$ be a conservative vector field along a smooth path C. Remember, this means that there exists a function f such that $\mathbf{F}(\boldsymbol{r}(t)) = \nabla f(\boldsymbol{r}(t))$ for all $t \in [a, b]$, where $\boldsymbol{r}(t)$ is any parametrization of C. If ∇f is continuous along C, then

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a)). \tag{10}$$

RE 5 Prove this theorem by reducing it to the Calculus 1 Evaluation Theorem.

Ex 6 Find the work done by gravity on a moving particle with mass m that travels from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along any piecewise-smooth curve C.

Recall that the gravitational field is given by

$$\vec{F}(\vec{x}) = \frac{-mMG}{\|\vec{x}\|^3}\vec{x}$$

and has potential function $f(\vec{x}) = \dfrac{mMG}{\|\vec{x}\|}$

By (10), we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,2,0) - f(3,4,12)$$

$$= \frac{mMG}{\sqrt{4+4}} - \frac{mMG}{\sqrt{9+16+144}} = mMG\left(\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{169}}\right)$$

$$= \boxed{mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)}$$

Independence of Path

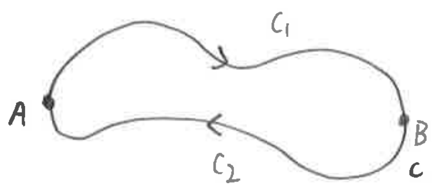
We did an example last class which showed us that path integrals depend on the paths you integrate along, not just the starting and ending points. However, equation (10) and the last **Ex** suggest something different.

Clearly, if $\mathbf{F}$ is a conservative vector field, then $\int_C \mathbf{F} \cdot dr$ does not depend on the path C. The situation appears to be slightly more general.

Theorem 7 *$\int_C \mathbf{F} \cdot dr$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot dr = 0$ for every closed path C in D.*

Ex 8 Proof:

A closed path is one whose intitial and terminal pt coincide



Choose any other pt and break the "loop" into two paths, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r} = 0 .$$

Conversely, if $\int_C \vec{F}\cdot d\vec{r} = 0$, then we reverse the argument to get

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}.$$

$\square$

As we mentioned before, we already know that conservative vector fields are independent of path, essentially by definition. The next theorem tells us that conservative vector fields are the *only* ones that are independent of path.

Theorem 9 *Suppose D is an open, connected region in $\mathbb{R}^n$. Let $\mathbf{F}$ be a continuous vector field on D. If $\int_C \mathbf{F} \cdot dr$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D;* i.e, *there is a function $f : D \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.*

connected = path-connected = any two points can be joined by a curve (or path) in D.

open means ~~any~~ every point has a disc around it that's contained in D.

RE 10 Read the proof: p. 745.

One question still remains: How can we tell if a vector field is conservative or not? The answer in (or on?) $\mathbb{R}^2$ comes in the form of two more theorems.

Theorem 11 *If $\mathbf{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

RE 12 Prove this by using Clairaut's Theorem.

Theorem 13 *Let $\mathbf{F} = P\boldsymbol{i} + Q\boldsymbol{j}$ be a vector field on an open, simply connected region D. Suppose that P and Q have continuous first-order derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D. Then $\mathbf{F}$ is conservative.

We'll prove this theorem in the next section.

Ex 14 Determine whether or not the vector field $\mathbf{F}(x, y) = (x - y)\boldsymbol{i} + (x - 2)\boldsymbol{j}$ is conservative.

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, so $\vec{F}$ is not conservative by Thm 11.

(really, the contrapositive to Thm 11).

Ex 15 Determine whether or not the vector field $\mathbf{F}(x,y) = (3+2xy)\boldsymbol{\imath} + (x^2-3y^2)\boldsymbol{\jmath}$ is conservative.

$$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x.$$

The domain of $\vec{F}$ is all of $\mathbb{R}^2$, which is simply connected.

Therefore, $\vec{F}$ is conservative by Thm 12.

Ex 16 (*a*) If $\mathbf{F}(x,y) = \langle 3+2xy, x^2-3y^2\rangle$, find a potential function f such that $\mathbf{F} = \nabla f$. (*b*) Evaluate the path integral $\int_C \mathbf{F}\cdot d\boldsymbol{r}$, where C is determined by $\boldsymbol{r}(t) = \langle e^t \sin t, e^t \cos t\rangle$, $0 \le t \le \pi$.

We need an f s.t. $D_x f = 3+2xy$ and $D_y f = x^2-3y^2$

$$\int D_x f\,dx = \int 3+2xy\,dx \qquad \int D_y f\,dy = \int x^2-3y^2\,dy$$

$$f = 3x + x^2y + g(y) \qquad f = x^2y - y^3 + h(x)$$

Thus, $\boxed{f = 3x + x^2y - y^3}$

then $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0))$

$$\vec{r}(\pi) = (0, -e^{\pi}), \quad \vec{r}(0) = (0,1)$$

so $\int_C \vec{F}\cdot d\vec{r} = -(-e^{\pi})^3 - (-1^3) = \boxed{e^{3\pi}+1}$

* This is much easier than actually evaluating the integral (the old way).

Conservation of Energy

Newton's Second Law says that $\mathbf{F} = m\boldsymbol{a}$. Remembering that $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t)$ and $\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t)$, we get a second order differential equation (or SODE):

$$\mathbf{F}(\boldsymbol{r}(t)) = m\ddot{\boldsymbol{r}}(t).$$

Let C be a path defined by the parametrized curve $\boldsymbol{r}(t)$, $t \in [a, b]$. Furthermore, let $\boldsymbol{r}(a) = A$ and $\boldsymbol{r}(b) = B$.

RE 17 Use this and the fact that kinetic energy is given by $K(\boldsymbol{r}(t)) = \frac{1}{2}m\|\boldsymbol{v}(t)\|^2$ to show that

$$W = \int_C \mathbf{F} \cdot d\boldsymbol{r} = K(B) - K(A).$$

RE 18 Suppose that $\mathbf{F}$ is a conservative vector field; *i.e.*, $\mathbf{F} = \nabla f$. The *potential energy* of an object at a point (x, y, z) is defined to be $P(x, y, z) = -f(x, y, z)$ (hence the name *potential function*). Therefore $\mathbf{F} = -\nabla P$. Show that

$$P(A) + K(A) = P(B) + K(B).$$

This is called the *Law of Conservation of Energy.*