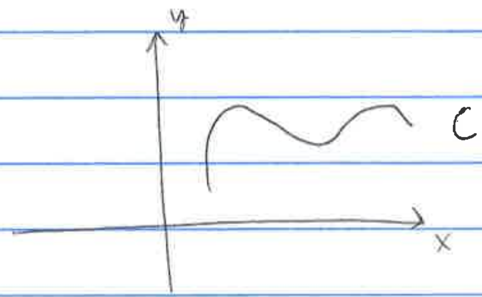
13.2 Line Integrals

Here, we want to take an integral similar to a single integral, but over a curve C (or γ) in the plane rather than just over an interval.



think of C as the curve given by the parametric equations

$$\vec{r}(t) = \langle x(t), y(t) \rangle \qquad a \le t \le b$$

and assume that $\vec{r}$ (or C) is smooth, i.e., $\dot{\vec{r}}$ is continuous and $\dot{\vec{r}}(t) \neq \vec{0}$.

If f is a smooth function defined on C, then the line integral of f along C is

$$\int_C f(x,y)\, ds = \lim_{\Delta s \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s$$

if the limit exists.

(and again in Ch. 10)

Recall from §9.2 that the arc length of C is

$$L = \int_a^b ds$$

$$L = \int_a^b \|\dot{\vec{r}}(t)\|\, dt \qquad \text{and} \qquad s(t) = \int_a^t \|\dot{\vec{r}}(u)\|\, du$$

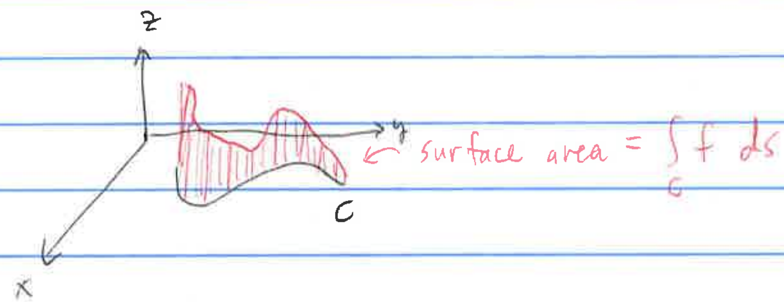
$$L = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt$$

A similar argument shows that

$$\int_C f(x,y)\,ds = \int_a^b f(x,y)\sqrt{(\dot{x})^2+(\dot{y})^2}\,dt$$

The line integral doesn't depend on the parametrization so long as the curve is only traversed once.

As with single integrals from Calc I (and beyond), we can think of line integrals of positive functions as areas over the curve C.



Ex. $\int_C (2+x^2y)\,ds$, C is the upper half of the unit circle $x^2+y^2=1$.

C is parametrized as $\vec{r}(t)=\langle \cos t, \sin t\rangle \quad t\in 0,\pi$

$\dot{\vec{r}}(t)=\langle -\sin t, \cos t\rangle$

$ds = \sqrt{\sin^2 t+\cos^2 t}\,dt = dt$

So

$$\int_C (2+x^2y)\,ds = \int_0^{\pi} 2+\cos^2 t\sin t\,dt$$

$$= \int_0^{\pi} 2\,dt - \int_1^{-1} u^2\,du$$

$$= 2\pi + \tfrac{1}{3}(2) = \boxed{2\pi+\tfrac{2}{3}}$$

(*) If a curve has two different pieces, then integrate over each piece separately.

Some times we will want to take the line integrals wrt x or y only, and not ds.

$$\int_C f(x,y)\,dx = \int_a^b f(x,y)\,\dot{x}(t)\,dt$$
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,\dot{y}(t)\,dt$$

Frequently we take both of these at the same time (the x and y integrals occur at the same time). Then we write:

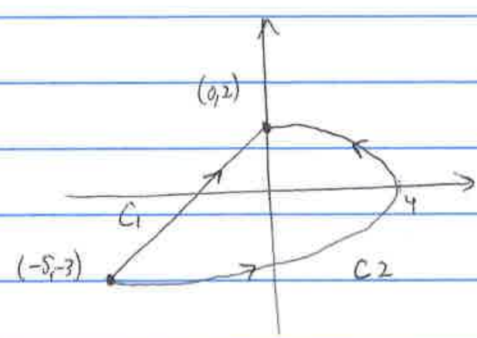
$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

Ex. $\int_C y^2\,dx + x\,dy$

where C is given by:

a) C_1 = line segment from $(-5,-3)$ to $(0,2)$

b) C_2 = segment of parabola $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$



a) C_1 is given by $x = 5t-5$ $\quad 0 \le t \le 1$

$y = 5t - 3$

(recall: line segments are given by $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \quad 0 \le t \le 1$)

$$\text{so } \int_{C_1} y^2\,dx + x\,dy = \int_0^1 (5t-3)^2\, 5\,dt + \int_0^1 (5t-5)\,5\,dt$$

$$= 5\int_0^1 25t^2 - 30t + 9\,dt + 5\int_0^1 5t - 5\,dt$$

$$= 5\int_0^1 25t^2 - 25t + 4\,dt$$

$$= \cdots = -5/6$$

b) Take y to be the parameter: $x = 4 - y^2$ $\quad y = y$ $\quad -3 \le y \le 2$

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4-y^2)\,dy$$

$$= \int_{-3}^{2} 4 - y^2 - 2y^3\,dy$$

$$= 4y - \frac{1}{3}y^3 - \frac{1}{2}y^4 \Big|_{-3}^{2} = \cdots = \frac{245}{6}$$

* Notice: These are <u>not</u> equal.

Path integrals depend on the path, in general!

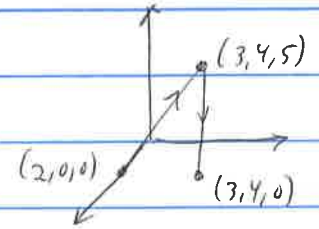
Also, reversing the direction you traverse the path will give you a negative sign (just like calc I).

<u>Path Integrals in Space:</u> $\quad C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad a \le t \le b$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x,y,z)\sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2}\,dt$$

$$= \int_a^b f(\vec{r})\,\|\dot{\vec{r}}\|\,dt$$

Ex. $\int_C y\,dx + z\,dy + x\,dz$



C_1: $\vec r(t) = (1-t)\langle 2,0,0\rangle + t\langle 3,4,5\rangle$

$= \langle 2+t, 4t, 5t\rangle \qquad 0 \le t \le 1$

C_2: $\vec r(t) = \langle 3, 4, 5-5t\rangle \qquad 0 \le t \le 1$

Do it.

Ex. $\int_C y \sin z \, ds \qquad C$ is the helix: $\vec r(t) = \langle \cos t, \sin t, t\rangle$

$0 \le t \le 2\pi$

Do it.

Line Integrals over vector Fields

Suppose $\vec F(x,y,z)$ is a continuous vector field on $\mathbb{R}^n$.
(think: force field.)

If $\vec T$ is the ~~an~~ unit tangent vector field along a curve C, then the work done by ~~a par~~ $\vec F$ on a particle moving along C is approximately $\vec F \cdot \vec T$ for small neighborhoods around each point. Add them all up; i.e., integrate.

So Work done by $\vec F = W = \int_C \vec F(x,y,z) \cdot \vec T(x,y,z)\, ds$

but $\vec T = \dfrac{\dot{\vec r}}{\|\dot{\vec r}\|}$ and $ds = \|\dot{\vec r}\|\, dt$ so this becomes

$$\int_C \vec F(x,y,z) \cdot \dot{\vec r}(x,y,z)\, dt \quad \text{or}$$

$$\boxed{\int_C \vec F(\vec r(t)) \cdot \dot{\vec r}(t)\, dt} \qquad \left\{ \begin{array}{l} \text{This is the line integral of} \\ \vec F \text{ along } C \end{array} \right.$$

$$= \int_C \vec F \cdot d\vec r$$

Ex. Find the work done by $\vec{F}(x,y) = \langle x^2, -xy \rangle$ on a particle moving along the quarter-circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$ $0 \le t \le \pi/2$

$$W = \int_C \vec{F} \cdot \underbrace{\dot{\vec{r}}\, dt}_{dr} = \int_0^{\pi/2} \langle \cos^2 t, -\cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t \, dt$$

$$= +2 \int_1^0 u^2 \, du = -2 \int_0^1 u^2 \, du = \boxed{-2/3}$$

Note: $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$ because $\vec{T}$ is replaced by $-\vec{T}$ along $-C$.

Ex. $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle xy, yz, zx \rangle$ and C is given by $\vec{r} = \langle t, t^2, t^3 \rangle$ $0 \le t \le 1$

So: $\dot{\vec{r}} = \langle 1, 2t, 3t^2 \rangle$

$\vec{F}(t) = \langle t^3, t^5, t^4 \rangle$

$\vec{F} \cdot \dot{\vec{r}} = t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 t^3 + 5t^6 \, dt = \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$

Finally, we notice a connection between path integrals of functions and vector fields:

$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

$\dot{\vec{r}} = \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j} + \dot{z}(t)\vec{k}$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left[P(x,y,z)\dot{x}(t) + Q(x,y,z)\dot{y}(t) + R(x,y,z)\dot{z}(t) \right] dt$$

$$= \int_C P\,dx + Q\,dy + R\,dz \; !$$