§ 12.8 : Change of Variables in Multiple Integrals

Recall from Calc I: If $x = g(u)$ w/ $a = g(c)$, $b = g(d)$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\, g'(u)\, du$$

$$= \int_c^d f(x(u))\, \frac{dx}{du}\, dx$$

* Change of variables formula.

We've also seen a version for double integrals

$$\iint_R f(x,y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

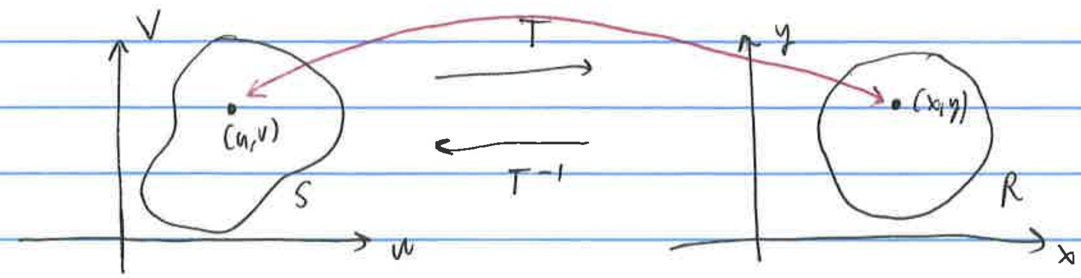
where S is the region in the $r\theta$-plane that corresponds to R in the xy-plane.

More generally, we want to consider a transformation

$$T(u,v) = (x,y)$$

from the uv-plane to the xy-plane, where x and y are related to u and v by

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases} \quad \text{or} \quad \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$



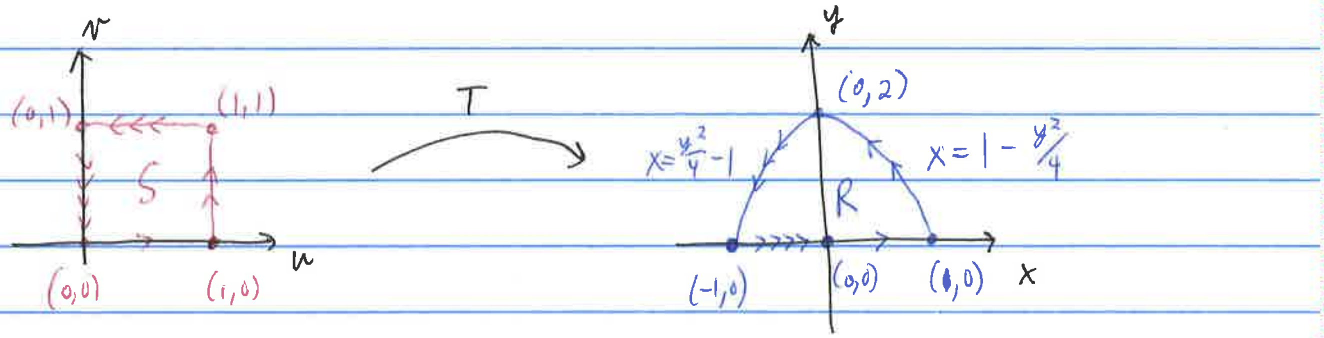
If T is one-to-one, then it may be possible to also write

$$\begin{cases} u = G(x,y) \\ v = H(x,y) \end{cases} \quad \text{or} \quad \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

Ex. A transformation T is defined by

$$x = u^2 - v^2$$
$$y = 2uv$$

Find the image of the square $S = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$



Do each side one at a time. For example,

$$\begin{cases} \rightarrow \text{ has } v = 0 \text{ and } 0 \le u \le 1 \\ \twoheadrightarrow \text{ has } u = 1 \text{ and } 0 \le v \le 1 \\ \ggg \text{ has } v = 1 \text{ and } 0 \le u \le 1 \\ \ggg\!\!\rightarrow \text{ has } u = 0 \text{ and } 0 \le v \le 1 \end{cases}$$

Then $\rightarrow$ becomes $y = 0,\ x : 0 \to 1$

$\twoheadrightarrow$ becomes $x = 1 - v^2$, $y = 2v$ so $x = 1 - \frac{y^2}{4}$

etc...

To use changes-of-variables in integrals we need:

<u>Def'n</u>. The <u>Jacobian</u> of the transformation $T:(u,v) \to (x,y)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Then $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv = dx\, dy$

* This, of course, assumes that the transformation T is differentiable in some sense (in <u>this</u> sense).

Without working out all of the details (sorry ⍨) we get the following formula for change of coordinates in a double integral:

<u>Thm</u>. If T is smooth, w/ Jacobian (nonzero) mapping a region S in the uv-plane onto a region R in the xy plane, f is continuous on R, S and R are Type I or II integrals, and T is one-to-one except possibly on the boundary of S, then

$$* \quad \boxed{\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.}$$

Ex. Verify that this works for polar coordinates:

$$\iint_R f(x,y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta}$$

$$= \cos\theta\,(r\cos\theta) - \sin\theta\,(-r\sin\theta)$$

$$= r\,(\cos^2\theta + \sin^2\theta)$$

$$= r$$

So this agrees! :)

Ex. (3) $x = \dfrac{u}{u+v}$, $y = \dfrac{u}{u-v}$ Find the Jacboin

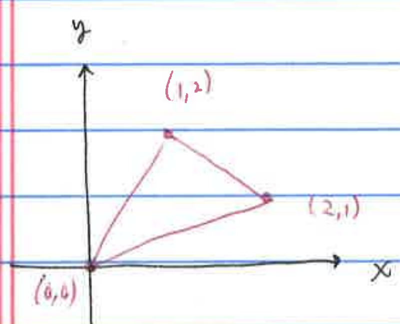
$$\frac{\partial x}{\partial u} = \frac{(u+v)-u}{(u+v)^2} = \frac{v}{(u+v)^2} \qquad \frac{\partial y}{\partial u} = \frac{(u-v)-u}{(u-v)^2} = \frac{-v}{(u-v)^2}$$

$$\frac{\partial x}{\partial v} = \frac{-u}{(u+v)^2} \qquad \frac{\partial y}{\partial v} = \frac{+u}{(u-v)^2}$$

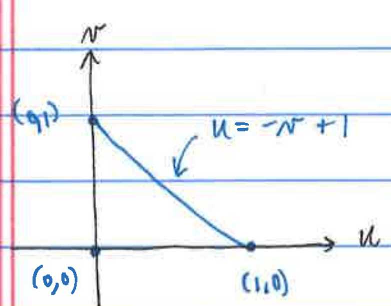
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{uv}{(u+v)^2(u-v)^2} - \frac{uv}{(u+v)^2(u-v)^2} = 0$$

Ex. (4) $\iint_R (x-3y)\, dA$ R is the triangle w/ vert $(0,0), (2,1), (1,2)$.

$x = 2u+v$, $y = u+2v$



$$\begin{aligned} 2x &= 4u + 2v \\ - \quad y &= u + 2v \\ \hline 2x - y &= 3u \end{aligned} \Rightarrow u = \frac{2}{3}x - \frac{1}{3}y$$



$$\begin{aligned} x &= 2u + v \\ - \quad 2y &= 2u + 4v \\ \hline x - 2y &= -3v \end{aligned} \Rightarrow v = \frac{2}{3}y - \frac{1}{3}x$$

$u(2,1) = \frac{2}{3}(2) - \frac{1}{3} = 1$

$v(2,1) = \frac{2}{3}(1) - \frac{2}{3} = 0$

$u(1,2) = \frac{2}{3} - \frac{2}{3} = 0$

$v(1,2) = \frac{2}{3}(2) - \frac{1}{3} = 1$

Jacobian: $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

$= 4 - 1 = 3$

so $dA = 3\, du\, dv = 3\, dv\, du$

then the integral is

$$\iint_R (x-3y)\, dA = \int_0^1 \int_0^{1-v} \Big[(2u+v)-3(u+2v)\Big]\, 3\, du\, dv$$

$$= \int_0^1 \int_0^{1-v} 3(-u-5v)\, du\, dv$$

$$= \int_0^1 \int_0^{1-v} -3u - 15v\, du\, dv$$

$$= \int_0^1 \left[-\frac{3}{2}u^2 - 15vu \right]_0^{1-v} dv$$

$$= \int_0^1 -\frac{3}{2}(1-v)^2 - 15v(1-v)\, dv$$

$$= \int_0^1 -\frac{3}{2}(1-2v+v^2) - 15v + 15v^2\, dv$$

$$= \int_0^1 \frac{27}{2}v^2 - 12v - \frac{3}{2}\, dv$$

$$= \left[\frac{9}{2}v^3 - 6v^2 - \frac{3}{2}v \right]_0^1$$

$$= \frac{9}{2} - 6 - 3/2 = \frac{9-12-3}{2} = \frac{-6}{2} = \boxed{-3}$$